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Renormalization Group and Asymptotic Spin-Charge Separation for Chiral Luttinger Liquids

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Abstract The phenomenon of *Spin-Charge separation* in non-Fermi liquids is well understood only in certain *solvable* $d = 1$ fermionic systems. In this paper we furnish the first example of *asymptotic* Spin-Charge separation in a $d = 1$ *nonsolvable* model. This goal is achieved using Renormalization Group approach combined with Ward-Identities and Schwinger-Dyson equations, corrected by the presence of a bandwidth cut-offs. Such methods, contrary to bosonization, could be in principle applied also to lattice or higher dimensional systems.

Keywords Renormalization group · Fermionic systems · Ward identities · Schwinger-Dyson equation · Spin-charge separation

1 Introduction and Main Results

In recent years the properties of non-Fermi liquids have been extensively investigated, both from experimental and theoretical point of view. In particular, one of the most spectacular feature appearing in non-Fermi liquids is the phenomenon of *Spin-Charge (SC) separation*, which is surely relevant for the physics of metals so anisotropic to be considered one dimensional, see for instance [17] or [11]. In addition, it is the key property in the Anderson theory of high-T_c superconductors (cuprates described by $d = 2$ fermionic systems), [1].

As it is well known, SC separation is an highly non-perturbative phenomenon, and its occurrence in fermionic models is quite hard to prove. Up to now it has been obtained only for the *spinning Luttinger model* (or *Mattis model*), [16], describing two kind of fermions, with spin $1/2$ and interacting through a short ranged potential. Its exact solvability is due to the linear dispersion relation (without any form of high energy cutoff) requiring a "Dirac sea" of fermions with negative energy; such features are quite unrealistic in a model aiming to describe conduction electrons in a metal, but they allow to map the interacting fermionic

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system into a non-interacting bosonic one, and to write the Hamiltonian as sum of two, decoupled Hamiltonians, respectively for spin and the charge degree of freedom. As a result, the two-point Schwinger function, in the case of local interaction, factorizes into the product of two functions, with different *Fermi velocities*, s_ρ, s_σ, and different *critical indices* η_ρ, η_σ, for the density (ρ) and the spin (σ) respectively:

$$S_\omega(x_0, x_1) = \frac{1}{(x_0 s_\sigma + i\omega x_1)^{1/2+\eta_\sigma}} \frac{1}{(x_0 s_\rho + i\omega x_1)^{1/2+\eta_\rho}}. \tag{1}$$

Such a factorization appears also in the n-point Schwinger functions (see [14] for an explicit formula), and it causes a phenomenology considerably different from the one of Fermi liquids [18].

For certain values of the parameters the spinning Luttinger model reduces to the *Chiral Luttinger* model; in such a case (1) still hold but $\eta_\rho = \eta_\sigma = 0$, that is in such a model only SC separation and no anomalous dimension is present.

The occurrence of SC separation in more realistic *non solvable* models, like the Hubbard model, has never been established, as a consequence of the fact that lattice or nonlinear bands prevents the use of bosonization. It is important to understand SC separation in the framework of Renormalization Group (RG), which is actually the only method which can be in principle applied in full generality to the complex models appearing in condensed matter in any dimension. However even in $d = 1$, in which RG methods have been extensively applied,—from the fundamental perturbative analysis in [17] to the non-perturbative and rigorous construction of Luttinger liquids in [2–4, 6, 15]—very few attention has been devoted to the application to SC separation effects (with the exception of the recent paper [9], in which however several approximations are introduced).

In this paper we will show that SC separation can be established in a *non exactly solvable* model by using RG methods; the model we consider is the *Chiral Luttinger liquid model* with a *bandwidth cut-off*, describing spinning fermions interacting through a short range potential. For physical applications, the presence of a *finite* momentum cut-off is essential as a linear dispersion relation can be a reasonable approximation for a non-relativistic dispersion relation only for momenta close to the Fermi surface; its presence prevents however the possibility of an exact solution through bosonization. This model have received a great deal of attention since the edge excitations in the fractional Quantum Hall effect are believed to be a physical realization of a Chiral Luttinger liquid [19].

1.1 Basic Definitions

We express the Chiral Luttinger liquid model directly in terms of *Grassmann variables*. Given the interval $[0, L]$, the inverse temperature β and a large integer, M, we introduce the lattice Λ_M made of the points $\mathbf{x} = (x_0, x_1) = (n_0 \frac{\beta}{M}, n_1 \frac{L}{M})$, for $n_0, n_1 = 0, 1, \dots M-1$. We also consider the lattice $\mathcal{D} = \mathcal{D}_L \times \mathcal{D}_\beta$ of points $\mathbf{k} = (k_0, k_1)$, with $k_0 = \frac{2\pi}{\beta}(n_0 + \frac{1}{2})$, $k_1 = \frac{2\pi}{L}(n_1 + \frac{1}{2})$, and $n_0, n_1 = 0, 1, \dots, M-1$. With each $\mathbf{k} \in \mathcal{D}$, we associate eight Grassmann variables, $\hat\psi^{(\leq N)\varepsilon}_{\mathbf{k},\omega,\sigma}$, for $\varepsilon, \omega, \sigma = \pm$: the label σ represents the spin of the field, and the index ω its chirality ('right' or 'left' moving particle). The Grassmann measure $P(d\psi^{(\leq N)})$ is defined in terms of the propagator, i.e. the covariance of the fields: $\langle \psi^{-\varepsilon(\leq N)}_{\mathbf{x},\omega,\sigma} \psi^{\varepsilon'(\leq N)}_{\mathbf{y},\omega',\sigma'} \rangle_0 = \varepsilon \delta_{\varepsilon,\varepsilon'} \delta_{\omega,\omega'} \delta_{\sigma,\sigma'} g^{(\leq N)}_\omega(\mathbf{x}-\mathbf{y})$ for

$$g^{(\leq N)}_\omega(\mathbf{x}-\mathbf{y}) \stackrel{def}{=} \frac{1}{\beta L} \sum_{\mathbf{k}\in\mathcal{D}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \frac{\hat\chi_N(\mathbf{k})}{-ik_0 + \omega k_1} \tag{2}$$

where $\hat{\chi}_N(\mathbf{k})$ is a smooth compact support function $\hat{\chi}_N(\mathbf{k}) \stackrel{def}{=} \hat{\chi}(\gamma^{-N}|\mathbf{k}|)$, where $\gamma > 1$ and $\hat{\chi}(t)$ is a $C_0^\infty(\mathbb{R}_+)$ such that

$$\hat{\chi}(t) \stackrel{def}{=} \begin{cases} 1 & \text{if } 0 \le t \le 1, \\ 0 & \text{if } t \ge \gamma. \end{cases} \tag{3}$$

For evaluating the Schwinger functions it is convenient to consider the *Generating Functional*, $\mathcal{W}(\varphi, J)$, defined in terms of the *external sources* J, φ and $\bar{\varphi}$ by the following *Grassmann integral*

$$e^{\mathcal{W}(\varphi,J)} \stackrel{def}{=} \int P(d\psi^{(\le N)}) \exp\left\{\lambda V(\psi^{(\le N)}) + \sum_{\omega,\sigma} \int d\mathbf{x}\, J_{\mathbf{x},\omega,\sigma} \psi^{+(\le N)}_{\mathbf{x},\omega,\sigma} \psi^{-(\le N)}_{\mathbf{x},\omega,\sigma}\right\}$$
$$\times \exp\left\{\sum_{\omega,\sigma} \int d\mathbf{x} [\varphi^+_{\mathbf{x},\omega,\sigma} \psi^{-(\le N)}_{\mathbf{x},\omega,\sigma} + \psi^{+(\le N)}_{\mathbf{x},\omega,\sigma} \varphi^-_{\mathbf{x},\omega,\sigma}]\right\} \tag{4}$$

where $\int d\mathbf{x} \stackrel{def}{=} \frac{\beta L}{M^2} \sum_{x_0, x_1 \in \Lambda_M}$ and, for $v(\mathbf{x})$ a smooth, rotation invariant, short range potential with $\hat{v}(0) = 1$,

$$V(\psi) = \sum_{\omega,\sigma} \int d\mathbf{x} d\mathbf{y} \psi^+_{\mathbf{x},\omega,\sigma} \psi^-_{\mathbf{x},\omega,\sigma} v(\mathbf{x} - \mathbf{y}) \psi^+_{\mathbf{y},\omega,-\sigma} \psi^-_{\mathbf{y},\omega,-\sigma}. \tag{5}$$

The propagator (2) describes fermions with a linear dispersion relation and a momentum cut-off selecting momenta $|\mathbf{k}| \le \gamma^{N+1}$, and $V(\psi)$ corresponds to a short range interaction involving only fermions with the *same* chirality.

The fields $J_{\mathbf{x},\omega,\sigma}$ are commuting variables, while the fields $\varphi_{\mathbf{x},\omega,\sigma}$, $\bar{\varphi}_{\mathbf{x},\omega,\sigma}$ are anticommuting. By taking $2n$ derivatives of $\mathcal{W}(\varphi, J)$ with respect to the $\varphi, \bar{\varphi}$ fields and m with respect the J field, and then putting $J = \varphi = \bar{\varphi} = 0$, one obtains the *Schwinger functions* at temperature β^{-1}, corresponding to $2n$ fermionic fields with m density insertions. As well known, the physical properties of the model can be deduced from the Schwinger functions, and an important role is played by the two-point Schwinger function, which is defined as

$$S_{N;\omega,\sigma}(\mathbf{x} - \mathbf{y}) = \frac{\partial^2 \mathcal{W}_N}{\partial \varphi^+_{\mathbf{x},\omega,\sigma} \partial \varphi^-_{\mathbf{y},\omega,\sigma}}(0, 0). \tag{6}$$

The lattice Λ_M is introduced just for technical reasons in order to avoid an infinite number of Grassmann variables, but our results are trivially uniform in M. The size L and the inverse temperature β plays the role of infrared cut-offs; one is interested in the physical quantities in the thermodynamic limit $L \to \infty$ and at low temperatures, that is up to $\beta = \infty$. We will prove the following result.

Theorem 1 *There exists $\varepsilon_0 > 0$ (N independent) such that, for $|\lambda| \le \varepsilon_0$, the limit of the two-points Schwinger function for $M, \beta, L \to \infty$ exists and has the form, for $\mathbf{x} \ne \mathbf{0}$*

$$S_{N;\omega,\sigma}(\mathbf{x}) = \frac{1}{(x_0 s + i\omega x_1)^{1/2} (x_0 s^{-1} + i\omega x_1)^{1/2}} [1 + R_N(\mathbf{x})] \tag{7}$$

with $R_N(\mathbf{x})$ bounded and such that

$$\lim_{|\mathbf{x}| \to \infty} R_N(\mathbf{x}) = 0 \quad \textit{and} \quad s = 1 + \frac{\lambda}{2\pi}. \tag{8}$$

The above theorem provides the first example of SC separation in a *non solvable* model. It is only *asymptotic*, that is up to terms which are negligible for large distances.

The proof of (7) is based on Renormalization Group methods combined with Ward Identities and Schwinger-Dyson equations, corrected by terms due to the presence of the momentum cut-offs which breaks the local symmetries. Hopefully the methods presented here could be applied to prove spin-charge separation in the $d = 1$ or even the $d = 2$ Hubbard model, despite such problems are of course much harder and pose several extra technical problems.

The rest of the paper is organized in the following way. In Sect. 2 and Sect. 3 we perform a Renormalization Group analysis; in the integration of the ultraviolet scales one has to improve the naive dimensional bounds taking advantage from the non-locality of the interaction, while in the infrared scales dramatic cancellations due to global phase symmetries are exploited. In Sect. 4 we bound the difference of the Schwinger functions with and without cut-offs, showing that it has a faster power law decay. Finally in Sect. 5 we implement Ward Identities and Schwinger-Dyson equations in the RG approach, obtaining an explicit expression of the Schwinger functions in the limit of removed cutoff.

2 Renormalization Group Analysis

We define the effective potential on scale N

$$\mathcal{V}^{(N)}(\psi^{(\leq N)}, \varphi, J) \stackrel{def}{=} \lambda V(\psi^{(\leq N)}) + \sum_{\omega,\sigma} \int d\mathbf{x} J_{\mathbf{x},\omega,\sigma} \psi^{+(\leq N)}_{\mathbf{x},\omega,\sigma} \psi^{-(\leq N)}_{\mathbf{x},\omega,\sigma} + \sum_{\omega,\sigma} \int d\mathbf{x} [\varphi^{+}_{\mathbf{x},\omega,\sigma} \psi^{-(\leq N)}_{\mathbf{x},\omega,\sigma} + \psi^{+(\leq N)}_{\mathbf{x},\omega,\sigma} \varphi^{-}_{\mathbf{x},\omega,\sigma}]. \tag{9}$$

Let $\hat{f}_h(\mathbf{k}) \stackrel{def}{=} \hat{\chi}(\gamma^{-h}|\mathbf{k}|) - \hat{\chi}(\gamma^{-(h-1)}|\mathbf{k}|)$. The RG analysis is triggered by the decomposition of $\hat{\chi}_N(\mathbf{k})$ as $\sum_{h=-\infty}^{N} \hat{f}_h(\mathbf{k})$, and correspondingly, the decomposition of the propagator, (2), as

$$g^{(\leq N)}_{\omega}(\mathbf{x}) = \sum_{h=-\infty}^{N} g^{(h)}_{\omega}(\mathbf{x}) \quad \text{for } g^{(h)}_{\omega}(\mathbf{x}) = \frac{1}{\beta L} \sum_{\mathbf{k}\in\mathcal{D}} e^{i\mathbf{k}\mathbf{x}} \frac{\hat{f}_h(\mathbf{k})}{-ik_0 + \omega k_1}. \tag{10}$$

Using standard techniques (see for instance [12], Appendix A2), for any positive integer q, there exists a constant C_q such that, for any $h \leq N$

$$|g^{(h)}_{\omega}(\mathbf{x})| \leq C_q \frac{\gamma^h}{1 + (\gamma^h |\mathbf{x}|)^q}. \tag{11}$$

From the basic properties of Grassman integrals it also follows that $\psi^{\varepsilon(\leq N)}_{\mathbf{x},\omega,\sigma} = \sum_{j=-\infty}^{N} \psi^{\varepsilon(j)}_{\mathbf{x},\omega,\sigma}$, where $\psi^{\varepsilon(j)}_{\mathbf{x},\omega,\sigma}$ is randomly independent from $\psi^{\varepsilon(i)}_{\mathbf{x},\omega,\sigma}$, for $i \neq j$; and has covariance $g^{(j)}_{\omega}(\mathbf{x})$. We then define the *effective potential on scale* k, $\mathcal{V}^{(k)}(\psi^{(\leq k)}, \varphi, J)$, such that

$$e^{\mathcal{V}^{(k)}(\psi^{(\leq k)},\varphi,J)} \stackrel{def}{=} \int P(d\psi^{[k+1,N]}) e^{\mathcal{V}^{(N)}(\psi^{[k+1,N]}+\psi^{(\leq k)},\varphi,J)} = e^{\sum_{n=1}^{\infty} \frac{1}{n!} \mathcal{E}^{T}_{k+1,N}(\mathcal{V}^{(N)};n)} \tag{12}$$

for $\psi^{\varepsilon[k,N]}_{\mathbf{x},\omega,\sigma} = \sum_{j=k}^{N} \psi^{\varepsilon(j)}_{\mathbf{x},\omega,\sigma}$ and $\psi^{\varepsilon(\leq k)}_{\mathbf{x},\omega,\sigma} = \sum_{j=-\infty}^{N} \psi^{\varepsilon(j)}_{\mathbf{x},\omega,\sigma}$; $\mathcal{E}^T_{k,N}$ is the *truncated expectation* with respect to the propagator $g^{[k,N]}_{\omega}(\mathbf{x})$:

$$\mathcal{E}^T_{k+1,N}(\mathcal{V}^{(N)}; n) \stackrel{def}{=} \mathcal{E}^T_{k+1,N}[\underbrace{\mathcal{V}^{(N)}|\cdots|\mathcal{V}^{(N)}}_{n \text{ times}}].$$

The effective potential is a polynomial of the fields. For $\varphi = 0$, (the case $\varphi \neq 0$ will be discussed in Sect. 4) we define the *kernels on scale* k, $W^{(n;2m)(k)}_{\omega,\underline{\sigma}}$, such that, for $\underline{\mathbf{z}} = \mathbf{z}_1, \ldots, \mathbf{z}_n$, $\underline{\mathbf{x}} = \mathbf{x}_1, \ldots, \mathbf{x}_m$, $\underline{\mathbf{y}} = \mathbf{y}_1, \ldots, \mathbf{y}_m$ and $\underline{\sigma} = \sigma'_1, \ldots \sigma'_n, \sigma_1, \ldots \sigma_m$, we have

$$\mathcal{V}^{(k)}(\psi^{(\leq k)}, 0, J) = \sum_{\substack{n\geq 0\\ m\geq 0}} \sum_{\substack{\underline{\sigma}',\underline{\sigma}\\ \omega}} \int d\underline{\mathbf{z}} d\underline{\mathbf{x}} d\underline{\mathbf{y}} \frac{W^{(n;2m)(k)}_{\omega,\underline{\sigma}}(\underline{\mathbf{z}}; \underline{\mathbf{x}}, \underline{\mathbf{y}})}{n!(2m)!}$$
$$\times \prod_{j=1}^{n} J_{\mathbf{z}_i,\omega,\sigma'_i} \prod_{i=1}^{m} \psi^{+(\leq k)}_{\mathbf{x}_i,\omega,\sigma_i} \psi^{-(\leq k)}_{\mathbf{y}_i,\omega,\sigma_i}. \tag{13}$$

As consequence of (12), the expression of the kernels in terms of the truncated expectations is:

$$W^{(n;2m)(k)}_{\omega,\underline{\sigma}}(\underline{\mathbf{z}}; \underline{\mathbf{x}}, \underline{\mathbf{y}})$$
$$= \prod_{i=1}^{n} \frac{\partial}{\partial J_{\mathbf{z}_i,\omega,\sigma'_i}}\bigg|_{J=0} \prod_{i=1}^{m} \frac{\partial}{\partial \psi^{+(\leq k)}_{\mathbf{x}_i,\omega,\sigma_i}} \frac{\partial}{\partial \psi^{-(\leq k)}_{\mathbf{y}_i,\omega,\sigma_i}}\bigg|_{\psi^{(\leq k)}=0}$$
$$\times \sum_{p=1}^{\infty} \frac{1}{p!} \mathcal{E}^T_{k+1,N}(\mathcal{V}^{(N)}(\psi^{(\leq k)} + \psi^{[k+1,N]}, J); p). \tag{14}$$

We introduce the following norm

$$\| W^{(n;2m)(k)}_{\omega,\underline{\sigma}} \|_k$$
$$\stackrel{def}{=} \frac{1}{L\beta} \int d\underline{\mathbf{x}} d\underline{\mathbf{y}} d\underline{\mathbf{x}}' d\underline{\mathbf{y}}' d\underline{\mathbf{z}} |\underline{\chi}_k(\underline{\mathbf{x}}' - \underline{\mathbf{x}}) \underline{\chi}_k(\underline{\mathbf{y}}' - \underline{\mathbf{y}}) W^{(n;2m)(k)}_{\omega,\underline{\sigma}}(\underline{\mathbf{z}}; \underline{\mathbf{x}}, \underline{\mathbf{y}})| \tag{15}$$

where $\underline{\chi}_k(\underline{\mathbf{x}}) = \prod_{j=1}^{n} \chi_k(\mathbf{x}_j)$ and $\chi_k(\mathbf{x})$ is the Fourier transform of $\sum_{j\leq k} \hat{f}_j(\mathbf{k})$.

We give more details on the truncated expectation of monomials of the fields; then, any polynomial can be computed by multilinearity. To shorten the notations we call

$$\psi_P = \prod_{f\in P} \psi^{-}_{\mathbf{x}(f),\omega,\sigma(f)} \psi^{+}_{\mathbf{y}(f),\omega,\sigma(f)} \tag{16}$$

where P is a set of labels. Given the *clusters of points* $P_1, \ldots, P_s$, the truncated expectation $\mathcal{E}^T_{k+1,N}[\psi_{P_1}|\cdots|\psi_{P_s}]$ is given by the sum of the values (with the relative sign) of all possible connected Feynman graphs, obtained representing graphically the monomial ψ_P as a set of oriented half lines coming out from the clusters of points and contracting them in all possible ways so that all the clusters are connected; to each line is associated a propagator $g^{[k+1,N]}_{\omega}$. Then the kernels $W^{(n;2m)(k)}_{\omega,\underline{\sigma}}$ can be written as sum over Feynman graphs as well, and the presence of cutoffs make each of them finite. Each connected Feynman graph made of p vertices is bounded by $C^p|\lambda|^p/p!$; anyway their number is $O(p!^2)$ so that the sum of the

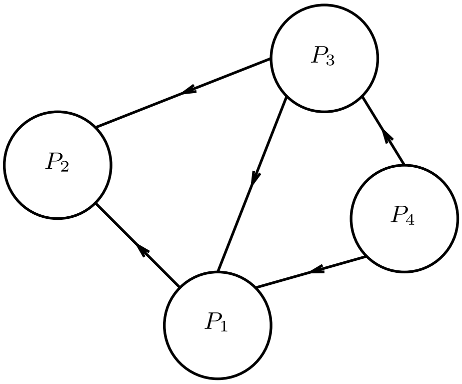


Fig. 1 An example of Feynman graph corresponding to one possible contribution to the truncated expectation of the clusters $P_1, \dots, P_4$. The lines with the arrows are the propagator: not all of them are necessary to connect the four clusters

graphs giving the truncated expectations are bounded by $C^p|\lambda|^p p!$, from which convergence of the series expansion in λ does not follow. The combinatorial bound can be improved using the idea in [7]: the anticommutativity of fermions produces dramatic cancellations among Feynman graphs, which are lost if the sum of graphs is simply bounded by the sum of their absolute values.

In order to exploit such cancellations it is then convenient to use a different representation of the truncated expectations: here we follow the standard technique of [10] and [8] (see also [13] and, for a detailed derivation, [12]).

$$\mathcal{E}^T_{k+1,N}[\psi_{P_1}|\cdots|\psi_{P_s}] = \sum_T \prod_{l\in T} g^{[k+1,N]}_\omega(\mathbf{x}_l - \mathbf{y}_l) \int dP_T(\mathbf{t}) \det G^T_{k+1,N}(\mathbf{t}) \tag{17}$$

where:

(1) T is a set of lines forming a *tree* between the clusters of points $P_1, \dots, P_s$, i.e. T is a set of lines which becomes a tree if all the points in the same cluster are identified; $n \stackrel{def}{=} \sum_{j=1}^s |P_j|$;
(2) $\mathbf{t} = \{t_{i,i'} \in [0,1], 1 \le i, i' \le s\}$ and $dP_T(\mathbf{t})$ is a probability measure with support on a set of $\mathbf{t}$ such that $t_{i,i'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i \in \mathbb{R}^s$ of unit norm;
(3) $G^T_{k+1,N}(\mathbf{t})$ is a $(n-s+1)\times(n-s+1)$ matrix, whose elements are given by

$$[G^T_{k+1,N}(\mathbf{t})]_{(j,i),(j',i')} = t_{j,j'} g^{[k+1,N]}_\omega(\mathbf{x}_{j,i} - \mathbf{x}_{j',i'}) \tag{18}$$

where $1 \le j, j' \le s$ and $1 \le i \le |P_j|$, $1 \le i' \le |P_{j'}|$, such that the lines $l = \mathbf{x}_{j,i} - \mathbf{x}_{j',i'}$ do not belong to T.

The kernels $W^{(n;2m)(k)}_{\omega,\underline{\sigma}}$ can be written as a convergent series in λ, as it is shown by the following lemma.

Lemma 1 *There exists $\varepsilon_{k,N}$ such that, for any λ such that $|\lambda| \le \varepsilon_{k,N}$, $W^{(n;2m)(k)}$ are analytic in λ.*

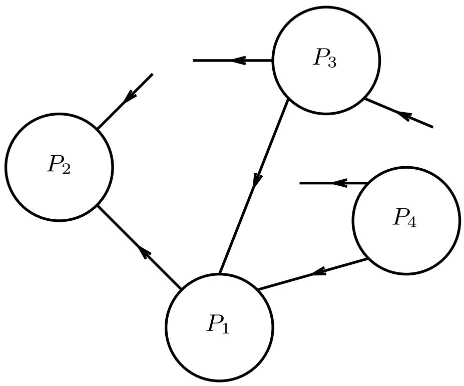


Fig. 2 Graphical representation of one term in (17). A tree graph connects the four clusters. The determinant correspond to contract the remaining half lines each other in all possible ways

Proof We bound the determinant $G^T_{k+1,N}(\mathbf{t})$ in (17) by using the *Gram-Hadamard inequality*: if A_i, B_j are vectors in a Hilbert space with scalar product $\langle\cdot,\cdot\rangle$, then

$$|\det_{i,j}\langle A_i, B_j\rangle| \le \prod_i \sqrt{\langle A_i, A_i\rangle}\sqrt{\langle B_i, B_i\rangle}. \tag{19}$$

Let $\mathcal{H} = \mathbb{R}^s \otimes \mathcal{H}_0$, where $\mathcal{H}_0$ is the Hilbert space of complex, squared summable functions, with scalar product

$$\langle F, G\rangle = \sum_{i=1}^4 \frac{1}{L\beta}\sum_{\mathbf{k}} \hat{F}_i^*(\mathbf{k})\hat{G}_i(\mathbf{k}). \tag{20}$$

Since $G^T_{k+1,N}(\mathbf{t})$ in (17) can be written as

$$\begin{aligned} G^T_{ij,i'j'}(\mathbf{t}) &= t_{i,i'} g_\omega^{[h+1,N]}(\mathbf{x}_{ij} - \mathbf{y}_{i'j'}) \\ &= \langle \mathbf{u}_i \otimes A_{\mathbf{x}_{ij},\omega}, \mathbf{u}_{i'} \otimes B_{\mathbf{x}_{i'j'},\omega}\rangle \end{aligned} \tag{21}$$

where $\mathbf{u}_i \in \mathbb{R}^s$, $i = 1, \dots, s$, are the vectors such that $t_{i,i'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$, and

$$\begin{aligned} A_{\mathbf{x},\omega} &= \frac{1}{L\beta}\sum_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}} \frac{\sqrt{\hat{\chi}_{k,N}(\mathbf{k})}}{k_0^2 + k^2}, \\ B_{\mathbf{x},\omega} &= -\frac{1}{L\beta}\sum_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}} \sqrt{\hat{\chi}_{k,N}(\mathbf{k})}(ik_0 + \omega k) \end{aligned} \tag{22}$$

so that

$$\langle A, A\rangle^{\frac{1}{2}} \le C\gamma^{N-2k}, \qquad \langle B, B\rangle^{\frac{1}{2}} \le C\gamma^{2N}, \tag{23}$$

we get

$$|\det G^T_{k+1,N}(\mathbf{t})| \le C^{(\sum_{i=1}^s |P_i|/2 - s + 1)N} \gamma^{(\sum_{i=1}^s |P_i|/2 - s+1)(N-k)}. \tag{24}$$

The number of trees T in (17) is bounded by $C^{\sum_i |P_i|} s!$, for a suitable constant C; by using (14) and (17), bounding the determinants by (11) and the integrations over coordinates by

$$\int d\mathbf{x}|g_\omega^{[h,N]}(\mathbf{x})| \le C\gamma^{-h}, \qquad \int d\mathbf{x}|v(\mathbf{x})| \le C \tag{25}$$

we get

$$\|W^{(n,2m)(k)}_{\omega,\underline{\sigma}}\|_k \le \sum_{p=1}^{\infty} |\lambda|^p C^p \gamma^{-p3(N-k)}\gamma^{m(p-3N)}\gamma^{-nk}\gamma^{3N+k} \tag{26}$$

and convergence follows for λ small enough. □

The above lemma says that the kernels $W^{(n,2m)(k)}_{\omega,\underline{\sigma}}$ are analytic in λ with an estimated radius of convergence which shrinks to zero when $|N-k|\to\infty$; we will see in the rest of this section how to improve the above bound to get convergence uniformly in $N-k$, by exploiting suitable cancellations in the series expansion.

It is convenient to introduce the directional derivative

$$\partial_\omega = \frac{1}{2}\left(i\frac{\partial}{\partial k_0} + \omega\frac{\partial}{\partial k}\right).$$

We will skip, sometimes, the label ω in the kernels. Calling $\hat{W}^{(n;2m)(k)}_{\omega,\underline{\sigma}}(\underline{\mathbf{p}};\underline{\mathbf{k}},\underline{\mathbf{q}})$ the Fourier transform of $W^{(n;2m)(k)}_{\omega,\underline{\sigma}}(\underline{\mathbf{z}};\underline{\mathbf{x}},\underline{\mathbf{y}})$, we have the following lemma.

Lemma 2 *For* $|\lambda|$ *small enough*,

$$\begin{aligned}&\hat{W}^{(0;4)(k)}_{\underline{\sigma}}(0) = \lambda\delta_{\sigma,-\sigma'}, \qquad \hat{W}^{(1;2)(k)}_{\underline{\sigma}}(0) = \delta_{\sigma,\sigma'},\\ &\hat{W}^{(0;2)(k)}_{\omega,\underline{\sigma}}(0) = (\partial_\omega \hat{W}^{(0;2)(k)}_{\omega,\underline{\sigma}})(0) = (\partial_{-\omega}\hat{W}^{(0;2)(k)}_{\omega,\underline{\sigma}})(0) = 0.\end{aligned} \tag{27}$$

Proof Because of Lemma 1, we can write the kernels as a convergent power series in λ: $\hat{W}^{(n;2m)(k)}_{\omega,\underline{\sigma}}(\underline{\mathbf{p}};\underline{\mathbf{k}},\underline{\mathbf{q}}) = \sum_{p\ge 0}\lambda^p \hat{W}^{(n;2m)(k)}_{p;\omega,\underline{\sigma}}(\underline{\mathbf{p}};\underline{\mathbf{k}},\underline{\mathbf{q}})$. For any integer $p\ge 1$, we define $R_p\mathbf{k}$ as the rotation of $\mathbf{k}$ of an angle $\frac{\pi}{2p}$:

$$\begin{pmatrix}(R_p\mathbf{k})_0\\(R_p\mathbf{k})_1\end{pmatrix} = \begin{pmatrix}\cos(\frac{\pi}{2p}) & -\sin(\frac{\pi}{2p})\\ \sin(\frac{\pi}{2p}) & \cos(\frac{\pi}{2p})\end{pmatrix}\begin{pmatrix}k_0\\k_1\end{pmatrix} \tag{28}$$

so that, by the explicit expression of $\hat{g}^{[k,N]}_\omega$, and since $\hat{v}$ was defined invariant under rotations,

$$\hat{g}^{[k,N]}_\omega(R_p\mathbf{k}) = e^{-i\omega\frac{\pi}{2p}}\hat{g}^{[k,N]}_\omega(\mathbf{k}), \qquad \hat{v}(R_p\mathbf{k}) = \hat{v}(\mathbf{k}). \tag{29}$$

Since $\hat{W}^{(0;4)(k)}_{p;\underline{\sigma}}(\underline{\mathbf{k}})$ is expressed by a sum over connected Feynman graphs obtained contracting $4p-4$ field (for such a kernel $p\ge 1$), we have

$$\hat{W}^{(0;4)(k)}_{p;\underline{\sigma}}(R_p\underline{\mathbf{k}}) = e^{-i\omega\pi(1-\frac{1}{p})}\hat{W}^{(0;4)(k)}_{p;\underline{\sigma}}(\underline{\mathbf{k}}), \tag{30}$$

which implies $\hat{W}^{(0;4)(k)}_{p;\underline{\sigma}}(0)=0$ for any $p\ge 2$; while, for $p=1$, $\hat{W}^{(0;4)(k)}_{p;\underline{\sigma}}(0)$ equals the coupling, $\lambda\delta_{\sigma,-\sigma'}$. In the same way $\hat{W}^{(1;2)(k)}_{p;\underline{\sigma}}(\underline{\mathbf{k}})$ is sum over Feynman graphs obtained contracting $4p$ fields (for $p\ge 0$); then

$$\hat{W}^{(1;2)(k)}_{p;\underline{\sigma}}(R_p\underline{\mathbf{k}}) = e^{-i\omega\pi}\hat{W}^{(1;2)(k)}_{p;\underline{\sigma}}(\underline{\mathbf{k}}) \tag{31}$$

and $\hat{W}^{(1;2)(k)}_{p;\underline{\sigma}}(0)=0$ for $p\geq 1$; while for $p=0$ $\hat{W}^{(1;2)(k)}_{0;\underline{\sigma}}(0)=\delta_{\sigma,\sigma'}$. We also find

$$\begin{aligned}
&\hat{W}^{(0;2)(k)}_{p;\sigma}(R_p\mathbf{k})=e^{-i\omega\pi(1-\frac{1}{2p})}\hat{W}^{(0;2)(k)}_{p;\sigma}(\mathbf{k}),\\
&(\partial_\omega\hat{W}^{(0;2)(k)}_{p;\omega,\sigma})(R_p\mathbf{k})=e^{-i\omega\pi}(\partial_\omega\hat{W}^{(0;2)(k)}_{p;\omega,\sigma})(\mathbf{k}),\\
&(\partial_{-\omega}\hat{W}^{(0;2)(k)}_{p;\omega,\sigma})(R_p\mathbf{k})=e^{-i\omega\pi(1-\frac{1}{p})}(\partial_{-\omega}\hat{W}^{(0;2)(k)}_{p;\omega,\sigma})(\mathbf{k}).
\end{aligned}\tag{32}$$

Since $p\geq 1$, and $\hat{W}^{(0;2)(k)}_{1;\omega,\sigma}(\mathbf{k})\equiv 0$ by explicit computation, (27) is proved. □

We start now the multiscale integration. Using (12), we find

$$\begin{aligned}
e^{\mathcal{W}_N(0,J)}&=\int P(d\psi^{(\leq N-1)})\int P(d\psi^{(N)})e^{\mathcal{V}^{(N)}(\psi^{(\leq N)},0,J)}\\
&=\int P(d\psi^{(\leq N-1)})e^{\mathcal{V}^{(N-1)}(\psi^{(\leq N-1)},J)}
\end{aligned}\tag{33}$$

where $\mathcal{V}^{(N-1)}(\psi^{(\leq N-1)},0,J)$ has the same form of (13). We introduce an $\mathcal{L}$-operation defined on the kernels in the following way

$$\begin{aligned}
&\mathcal{L}\hat{W}^{(n;2m)(N-1)}_{\omega,\underline{\sigma}}(\underline{\mathbf{k}})=0\quad \text{if } n+m>2,\\
&\mathcal{L}\hat{W}^{(n;2m)(N-1)}_{\omega,\underline{\sigma}}(\underline{\mathbf{k}})=\hat{W}^{(n;2m)(N-1)}_{\omega,\underline{\sigma}}(\underline{\mathbf{k}})\quad \text{if } n+m\leq 2.
\end{aligned}\tag{34}$$

Then we can write

$$\begin{aligned}
e^{\mathcal{W}_N(0,J)}&=\int P(d\psi^{(\leq N-2)})\\
&\quad\times\int P(d\psi^{(N-1)})e^{\mathcal{L}\mathcal{V}^{(N-1)}(\psi^{(\leq N-1)},0,J)+\mathcal{R}\mathcal{V}^{(N-1)}(\psi^{(\leq N-1)},0,J)}
\end{aligned}\tag{35}$$

and integrating we arrive to an expression similar to the r.h.s. of (33) with $N-1$ replaced by $N-2$; and so on for the integration of the $\psi^{(k+1)}$ field. The above definition of $\mathcal{L}$ remains the same until the scale $k=0$. For the fields on scales $k<0$ we define:

$$\begin{aligned}
&\mathcal{L}\hat{W}^{(0;4)(k)}_{\underline{\sigma}}(\underline{\mathbf{k}})\stackrel{def}{=}\hat{W}^{(0;4)(k)}_{\underline{\sigma}}(0),\\
&\mathcal{L}\hat{W}^{(1;2)(k)}_{\underline{\sigma}}(\mathbf{p};\mathbf{k})\stackrel{def}{=}\hat{W}^{(1;2)(k)}_{\underline{\sigma}}(0;0),\\
&\mathcal{L}\hat{W}^{(0;2)(k)}_{\omega,\sigma}(\mathbf{k})\stackrel{def}{=}\hat{W}^{(0;2)(k)}_{\omega,\sigma}(0)+\mathbf{k}\partial_{\mathbf{k}}\hat{W}^{(0;2)(k)}_{\omega,\sigma}(0).
\end{aligned}\tag{36}$$

By Lemma 2, since $\mathbf{k}\partial_{\mathbf{k}}=\sum_{\omega'}D_{\omega'}(\mathbf{k})\partial_{\omega'}$, we have that

$$\begin{aligned}
&\mathcal{L}\hat{W}^{(0;4)(k)}_{\underline{\sigma}}(\mathbf{k},\mathbf{p},\mathbf{q})=\lambda\delta_{\sigma,-\sigma'},\qquad \mathcal{L}\hat{W}^{(1;2)(k)}_{\underline{\sigma}}(\mathbf{p};\mathbf{k})=\delta_{\sigma,\sigma'},\\
&\mathcal{L}\hat{W}^{(0;2)(k)}_{\omega,\sigma}(\mathbf{k})=0.
\end{aligned}\tag{37}$$

In performing the bounds, it is necessary to pass to the coordinate representation; for $0\leq k\leq N$, we define $\lambda_{k;\omega,\underline{\sigma}}(\underline{\mathbf{x}})$, $\nu_{k;\omega,\sigma}(\underline{\mathbf{x}})$ and $Z_{k;\omega,\underline{\sigma}}(\mathbf{z};\underline{\mathbf{x}})$ such that

$$\mathcal{L}\mathcal{V}^{(k)}(\psi,0,J)=\sum_{\omega,\underline{\sigma}}\int d\underline{\mathbf{x}}\lambda_{k;\omega,\underline{\sigma}}(\underline{\mathbf{x}})\psi^+_{\mathbf{x}_1,\omega,\sigma}\psi^-_{\mathbf{x}_2,\omega,\sigma}\psi^+_{\mathbf{x}_3,\omega,\sigma'}\psi^-_{\mathbf{x}_4,\omega,\sigma'}$$

$$+\sum_{\omega,\sigma}\int d\underline{\mathbf{x}}\gamma^k \nu_{k;\omega,\sigma}(\underline{\mathbf{x}})\psi^+_{\mathbf{x}_1,\omega,\sigma}\psi^-_{\mathbf{x}_2,\omega,\sigma}$$

$$+\sum_{\omega,\underline{\sigma}}\int d\mathbf{z}d\underline{\mathbf{x}}Z_{k;\omega,\underline{\sigma}}(\mathbf{z};\underline{\mathbf{x}})J_{\mathbf{z},\omega,\sigma}\psi^+_{\mathbf{x}_1,\omega,\sigma'}\psi^-_{\mathbf{x}_2,\omega,\sigma'} \tag{38}$$

while for $k<0$

$$\mathcal{L}\mathcal{V}^{(k)}(\psi,0,J)=\lambda\sum_{\underline{\sigma},\omega}\int d\underline{\mathbf{x}}\delta_{\sigma,-\sigma'}\delta_3(\underline{\mathbf{x}})\psi^+_{\mathbf{x}_1,\omega,\sigma}\psi^-_{\mathbf{x}_2,\omega,\sigma}\psi^+_{\mathbf{x}_3,\omega,\sigma'}\psi^-_{\mathbf{x}_4,\omega,\sigma'}$$

$$+\sum_{\omega,\underline{\sigma}}\int d\mathbf{z}d\underline{\mathbf{x}}\delta_{\sigma,\sigma'}\delta_2(\mathbf{z},\underline{\mathbf{x}})J_{\mathbf{z},\omega,\sigma}\psi^+_{\mathbf{x}_1,\omega,\sigma'}\psi^-_{\mathbf{x}_2,\omega,\sigma'} \tag{39}$$

where $\delta_3(\underline{\mathbf{x}})\stackrel{def}{=}\delta(\mathbf{x}_1-\mathbf{x}_2)\delta(\mathbf{x}_2-\mathbf{x}_3)\delta(\mathbf{x}_3-\mathbf{x}_4)$ while $\delta_2(\mathbf{z},\underline{\mathbf{x}})\stackrel{def}{=}\delta(\mathbf{x}_1-\mathbf{x}_2)\delta(\mathbf{x}_2-\mathbf{z})$. To have a uniform notation we will also use the definitions, for $k<0$, $\lambda_{k;\omega,\underline{\sigma}}(\underline{\mathbf{x}})\stackrel{def}{=}\lambda\delta_{\sigma,-\sigma'}\delta_3(\underline{\mathbf{x}})$ and $Z_{k;\omega,\underline{\sigma}}(\mathbf{z};\underline{\mathbf{x}})\stackrel{def}{=}\delta_{\sigma,\sigma'}\delta_2(\mathbf{z},\underline{\mathbf{x}})$.

It is well known, see for instance [2], that $V^{(k)}(\psi^{(\le k)},0,J)$ can be represented as a sum over *Gallavotti-Nicolò trees* (in the following simply called trees) defined in the following way.

The trees which can be constructed by joining a point r, the *root*, with an ordered set of $n\ge 1$ points, the *endpoints* of the tree, so that r is not a branching point. n will be called the *order* of the unlabeled tree and the branching points will be called the *non trivial vertices*. We associate a label $h\le N-1$ with the root, r and we introduce a family of vertical lines, labeled by an integer taking values in $[h,N+1]$, and we represent any tree $\tau\in\mathcal{T}_{h,n}$ so that, if v is an endpoint or a non trivial vertex, it is contained in a vertical line with index $h_v>h$, to be called the *scale* of v, while the root is on the line with index h. The tree will intersect the vertical lines in set of points different from the root and the endpoints; these points will be called *trivial vertices*. The set of the *vertices* of τ will be the union of the endpoints, the trivial vertices and the non trivial vertices. Note that, if v_1 and v_2 are two vertices and $v_1<v_2$, then $h_{v_1}<h_{v_2}$. Moreover, there is only one vertex immediately following the root, which will be denoted v_0 and can not be an endpoint; its scale is $h+1$. There is the constraint, for the end-points of scale h_v, that $h_v=h_{v'}+1$, if v' is the first non trivial vertex immediately preceding v. With each normal endpoint of scale h_v we associate $\mathcal{L}\mathcal{V}^{h_v-1}$ given by (34) if $h_v\ge 0$ or (36) if $h_v<0$.

We introduce a *field label* f to distinguish the field variables appearing in the terms $\mathcal{V}$ associated with the endpoints. If v is a vertex of the tree τ, P_v is a set of labels which

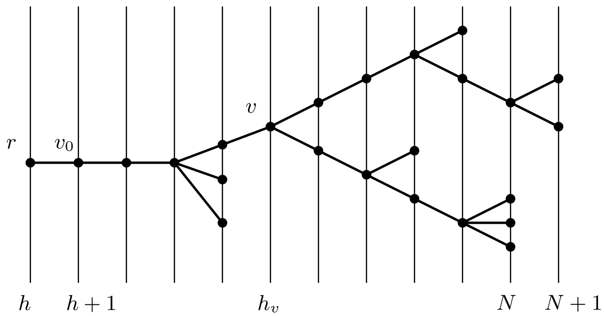


Fig. 3 A example of the Gallavotti-Nicolò tree

distinguish the *external fields of* v, that is the field variables of type ψ which belong to one of the endpoints following v and are not yet contracted in the vertex v. We will also call $n_v^\psi \stackrel{def}{=} |P_v|$ the number of such fields ψ, and n_v^J the number of the field variables of type J which belong to one of the endpoints following v. Finally, if v is not an endpoint, $\mathbf{x}_v$ is the family of all space-time points associated with one of the endpoints following v. It is easy to verify that

$$V^{(k)}(\psi^{(\leq k)}, 0, J) + \beta L E_k = \sum_{n\geq 1} \sum_{\tau \in \mathcal{T}_{k,n}} V^{(k)}(\tau) \tag{40}$$

where, if v_0 is the first vertex of τ and $\tau_1, \dots, \tau_s$ ($s = s_{v_0}$) are the subtrees of τ with root v_0, $V^{(k)}(\tau)$ is defined inductively by the relation, $k \leq N-1$

$$V^{(k)}(\tau) = \frac{(-1)^{s+1}}{s!} \mathcal{E}_{k+1}^T [\bar{V}^{(k+1)}(\tau_1)| \cdots | \bar{V}^{(k+1)}(\tau_s)] \tag{41}$$

where $\bar{V}^{(k+1)}(\tau) = \mathcal{R} V^{(k+1)}(\tau)$, for $\mathcal{R} = 1 - \mathcal{L}$, if the subtree τ_i contains more then one endpoint; if τ_i contains only one endpoint $\bar{V}^{(k+1)}(\tau)$ is equal to one of the terms in $\mathcal{L}\mathcal{V}^{h_v - 1}$.

With these definitions, we can rewrite $\mathcal{V}^{(k)}(\tau, \psi^{(\leq k)})$ as:

$$\begin{aligned} \mathcal{V}^{(k)}(\tau) &= \sum_{\mathbf{P} \in \mathcal{P}_\tau} \mathcal{V}^{(k)}(\tau, \mathbf{P}), \\ \mathcal{V}^{(k)}(\tau, \mathbf{P}) &= \int d\mathbf{x}_{v_0} \psi_{P_{v_0}}^{(\leq k)} K_{\tau,\mathbf{P}}^{(k)}(\mathbf{x}_{v_0}) \end{aligned} \tag{42}$$

where $K_{\tau,\mathbf{P}}^{(h+1)}(\mathbf{x}_{v_0})$ is defined inductively by (41).

By Lemma 1 and calling $\varepsilon_k = \max_{\omega,\underline{\sigma}} \max_{k\leq h\leq N} \{\|\lambda_{h;\omega,\underline{\sigma}}\|_k, \|\nu_{h;\sigma}\|_k\}$

$$\|K_{\tau,\mathbf{P}}^{(k)}\|_k \leq (c\varepsilon_{k+1})^{n-n_{v_0}^J} \gamma^{k(2-\frac{|P_{v_0}|}{2} - n_{v_0}^J)} \prod_{v \text{ not e.p.}} \gamma^{-(\frac{|P_v|}{2} - 2 + z_v + n_v^J)} \tag{43}$$

where, if $h_v > 0$, $z_v \equiv 0$. If $h_v \leq 0$, $z_v = 2$ if $|P_v| + 2n_v^J = 2$; $z_v = 1$ if $|P_v| + 2n_v^J = 4$, and $z_v = 0$ otherwise. The proof of (43) is an immediate consequence of the analysis in Sect. 3 of [2], based on (17) and the Gram-Hadamard inequality. The following lemma is an immediate consequence of the above bound.

Lemma 3 *There exist* $C > 1$ *and* $\varepsilon > 0$ *such that, for* $\varepsilon_{k+1} \leq \varepsilon$ *and* $\max_{h\geq k+1} \|Z_{k,\omega,\sigma}\|_k < 2$,

$$\|W_{\underline{\sigma},\underline{\omega}}^{(n;2m)(k)}\|_k \leq C^{n+m-1} \varepsilon^{(m-1\wedge 0)} \gamma^{k(2-n-m)} \tag{44}$$

for $(m \wedge 0) \stackrel{def}{=} \max\{m, 0\}$.

Proof For $h_v > 0$ the definition of $\mathcal{R}$ imposes the constraint that there are no v such that $(|P_v|, n_v^J) = (4,0), (2,0), (2,1)$; this implies that, for any v,

$$d_v \stackrel{def}{=} \frac{|P_v|}{2} - 2 + z_v + n_v^J > 0. \tag{45}$$

In order to sum over τ and $\mathbf{P}$ (for more details, see again [2]) we note that the number of unlabeled trees is $\leq 4^n$; fixed an unlabeled tree, the number of terms in the sum over the various labels of the tree is bounded by C^n, except the sums over the scale labels and the sets $\mathbf{P}$. Let $V(\tau)$ the nontrivial vertices of τ. In order to bound the sums over the scale labels and $\mathbf{P}$ we use the inequality

$$\prod_{v \text{ not e.p.}} \gamma^{-(\frac{|P_v|}{2}-2+z_v+n_v^J)} = \prod_{v\in V(\tau)} \gamma^{-(h_v-h_{v'})d_v} \leq \left(\prod_{v\in V(\tau)} \gamma^{-\frac{1}{40}(h_v-h_{v'})}\right) \prod_{v\in V(\tau)} \gamma^{-\frac{|P_v|}{40}} \tag{46}$$

and the first factor in the r.h.s. allow to bound the sums over the scale labels by C^n, while the sum over $\mathbf{P}$ can be bounded by using the following combinatorial inequality. Let $\{p_v, v\in\tau\}$ a set of integers such that $p_v \leq \sum_{i=1}^{s_v} p_{v_i}$ for all $v\in\tau$ which are not endpoints; then

$$\sum_{\mathbf{P}} \prod_{v\in V(\tau)} \gamma^{-\frac{|P_v|}{40}} \leq \prod_{v\in V(\tau)} \sum_{p_v} \gamma^{-\frac{p_v}{40}} B\left(\sum_{i=1}^{s_v} p_v, p_v\right) \leq C^n \tag{47}$$

where $B(n,m)$ is the binomial coefficient. □

3 Power Counting Improvement

The bound (44) is of course not sufficient to prove the boundedness of the kernels $W^{(n;2m)(k)}_{\underline{\sigma},\underline{\omega}}$, as we need to prove that $\bar{\varepsilon}_k$ is small uniformly in k. On the other hand $\mathbf{v}_h = (\lambda_h, \gamma^h \nu_h, Z_h^{(2)})$ verify the equation, for $h \geq 0$

$$\mathbf{v}_{h-1} = \mathbf{v}_h + \boldsymbol{\beta}_h(\mathbf{v}_h, \ldots, \mathbf{v}_N) \tag{48}$$

where $\boldsymbol{\beta}_h$ is expressed by a sum of trees such that the first nontrivial vertex has scale $h+1$ (from the property $\mathcal{LR}=0$), and $\mathbf{v}_N = (\lambda\delta_{-\sigma',\sigma}, 0, \delta_{\sigma,\sigma'})$. Iterating the above equation one finds

$$\begin{aligned} &\lambda_{h;\omega,\underline{\sigma}}(\underline{\mathbf{x}}) = W^{(0;4)(h)}_{\omega,\underline{\sigma}}(\underline{\mathbf{x}}),\\ &\gamma^h \nu_{h;\omega,\sigma}(\underline{\mathbf{x}}) = W^{(0;2)(h)}_{\omega,\sigma}(\underline{\mathbf{x}}), \qquad Z_{h;\omega,\underline{\sigma}}(\mathbf{z};\underline{\mathbf{x}}) = W^{(1;2)(h)}_{\omega,\underline{\sigma}}(\mathbf{z};\underline{\mathbf{x}}) \end{aligned} \tag{49}$$

and there is no reason a priori for which $\mathbf{v}_h$ should remain close to $\mathbf{v}_N$; this property will be established by a careful analysis implying an improvement of the previous bounds. We will prove in fact the following theorem.

Theorem 2 *For $|\lambda|$ small enough, there exist a constant $C_1 > 1$ such that, for $0 \leq h \leq N$*

$$\begin{aligned} &\|W^{(0;2)(h)}_{\sigma}\|_h \leq C_1|\lambda|\gamma^{-h}, \qquad \|W^{(1;2)(h)}_{\sigma',\sigma} - \delta_2\delta_{\sigma,\sigma'}\|_h \leq C_1|\lambda|\gamma^{-h},\\ &\|W^{(0;4)(h)}_{\sigma,\sigma'} - v\lambda\delta_2\delta_{\sigma,-\sigma'}\|_h \leq C_1|\lambda|\gamma^{-h}; \end{aligned} \tag{50}$$

where (with slight abuse of notation) $v\delta_2 \equiv \delta(\mathbf{x}-\mathbf{y})v(\mathbf{x}-\mathbf{u})\delta(\mathbf{u}-\mathbf{v})$.

An immediate consequence of the above theorem, together with (36), (37), (44), (49) is the boundedness of the kernels $W^{(n;2m)(k)}_{\omega,\underline{\sigma}}$ for $|\lambda|$ small enough (and since, for $h \geq 0$, $\gamma^{-h} \leq 1$)

$$\|W^{(n;2m)(k)}_{\underline{\sigma},\underline{\omega}}\|_k \leq C^{n+m-1}|C_1\lambda|^{(m-1\wedge 0)}\gamma^{k(2-n-m)}. \tag{51}$$

Proof The proof is by induction: we assume that (50) holds for $h : k+1 \le h \le N$; hence the hypothesis of lemma 3 are satisfied and we can use (44) to prove (50) for $h = k$.

To shorten the notation, in this proof we call $\eta \stackrel{def}{=} \psi^{\le k}$. By definition of the effective interaction, $\mathcal{V}^{(k)}$, we have

$$W^{(n;2m)(k)}_{\underline{\omega},\underline{\sigma}}(\underline{\mathbf{z}};\underline{\mathbf{x}},\underline{\mathbf{y}}) = \frac{\partial^{n+2m}\mathcal{V}^{(k)}}{\partial J_{\mathbf{z}_1,\sigma_1}\cdots\partial J_{\mathbf{z}_n,\sigma_n}\partial\eta^+_{\mathbf{x}_1,\omega_1}\partial\eta^-_{\mathbf{y}_1,\omega_1}\partial\eta^+_{\mathbf{x}_m,\omega_m}\partial\eta^-_{\mathbf{y}_m,\omega_m}}(0,0,0). \tag{52}$$

By the explicit expression of the function $\mathcal{V}^{(N)}$ we obtain:

$$\begin{aligned}\frac{\partial\mathcal{V}^{(k)}}{\partial\eta^+_{\mathbf{x},\omega,\sigma}}(\eta,J,0) &= J_{\mathbf{x},\omega,\sigma}\frac{\partial\mathcal{V}^{(k)}}{\partial\varphi^+_{\mathbf{x},\omega,\sigma}}(\eta,J,0)\\ &\quad + \lambda\int d\mathbf{w}v(\mathbf{x}-\mathbf{w})\left[\frac{\partial^2\mathcal{V}^{(k)}}{\partial J_{\mathbf{w},\omega,-\sigma}\partial\varphi^+_{\mathbf{x},\sigma}} + \frac{\partial\mathcal{V}^{(k)}}{\partial J_{\mathbf{w},\omega,-\sigma}}\frac{\partial\mathcal{V}^{(k)}}{\partial\varphi^+_{\mathbf{x},\omega,\sigma}}\right](\eta,J,0).\end{aligned} \tag{53}$$

Moreover the *Wick theorem* for Gaussian mean values gives

$$\begin{aligned}&\int P(d\psi^{[k+1,N]})\psi^{[k+1,N]-}_{\mathbf{x},\omega,\sigma}F(\psi^{[k+1,N]})\\ &\quad = \int d\mathbf{u}g^{[k+1,N]}_{\omega}(\mathbf{x}-\mathbf{u})\int P(d\psi^{[k+1,N]})\frac{\partial F}{\partial\psi^+_{\mathbf{u},\omega,\sigma}}(\psi^{[k+1,N]})\end{aligned} \tag{54}$$

for F any polynomial in the field. As direct application, we obtain

$$\begin{aligned}\frac{\partial\mathcal{V}^{(k)}}{\partial\varphi^+_{\mathbf{x},\omega,\sigma}}(\eta,J,\varphi) &= e^{-\mathcal{V}^{(k)}(\eta,J,\varphi)}\frac{\partial e^{\mathcal{V}^{(k)}(\eta,J,\varphi)}}{\partial\varphi^+_{\mathbf{x},\omega,\sigma}}\\ &= e^{-\mathcal{V}^{(k)}(\eta,J,\varphi)}\int P(d\psi^{[k+1,N]})(\psi^{[k+1,N]-}_{\mathbf{x},\omega,\sigma}+\eta^-_{\mathbf{x},\omega,\sigma})e^{\mathcal{V}^{(N)}(\psi+\eta,J,\varphi)}\\ &= \eta^-_{\mathbf{x},\omega} + \int d\mathbf{u}\; g^{[k+1,N]}_{\omega}(\mathbf{x}-\mathbf{u})\frac{\partial\mathcal{V}^{(k)}}{\partial\eta^+_{\mathbf{u},\omega,\sigma}}(\eta,J,\varphi).\end{aligned} \tag{55}$$

Another useful consequence is (since $g_\omega(0) = 0$):

$$\begin{aligned}\frac{\partial\mathcal{V}^{(k)}}{\partial J_{\mathbf{x},\omega,\sigma}}(\eta,J,\varphi) &= \eta^+_{\mathbf{x},\omega,\sigma}\eta^-_{\mathbf{x},\omega,\sigma}\\ &\quad + \int d\mathbf{u}g^{[k+1,N]}_{\omega}(\mathbf{x}-\mathbf{u})\left[\frac{\partial\mathcal{V}^{(k)}}{\partial\eta^-_{\mathbf{u},\omega,\sigma}}\eta^-_{\mathbf{x},\omega,\sigma} + \eta^+_{\mathbf{x},\omega,\sigma}\frac{\partial\mathcal{V}^{(k)}}{\partial\eta^+_{\mathbf{u},\omega,\sigma}}\right]\\ &\quad + \int d\mathbf{u}d\mathbf{u}'g^{[k+1,N]}_{\omega}(\mathbf{x}-\mathbf{u})g^{[k+1,N]}_{\omega}(\mathbf{x}-\mathbf{u}')\end{aligned}$$

$$\times \left[\frac{\partial^2 \mathcal{V}^{(k)}}{\partial \eta^+_{\mathbf{u},\omega,\sigma} \partial \eta^-_{\mathbf{u}',\omega,\sigma}} + \frac{\partial \mathcal{V}^{(k)}}{\partial \eta^+_{\mathbf{u},\omega,\sigma}} \frac{\partial \mathcal{V}^{(k)}}{\partial \eta^-_{\mathbf{u}',\omega,\sigma}} \right]. \tag{56}$$

We will use the following straightforward bounds, for $c_0, c_1, c_2 > 1$:

$$\begin{aligned} |g_\omega^{(h)}|_0 &\stackrel{def}{=} \sup_{\mathbf{x}} |g_\omega^{(h)}(\mathbf{x})| \le c_0 \gamma^h, \\ |g_\omega^{(h)}|_1 &\stackrel{def}{=} \int d\mathbf{x} |g_\omega^{(h)}(\mathbf{x})| \le c_1 \gamma^{-h}, \\ \int d\mathbf{x} |x_j| |g_\omega^{(h)}(\mathbf{x})| &\le c_2 \gamma^{-2h}. \end{aligned} \tag{57}$$

We start the improvement of the dimensional bounds by considering $W_\sigma^{(0;2)(k)}$. By symmetry we have $W_{-\sigma}^{(1;0)(k)}(\mathbf{w}) \equiv 0$; hence from (53) and (55) we expand the two-points kernel as in Fig. 4.

$$\begin{aligned} & W_\sigma^{(0;2)(k)}(\mathbf{x}, \mathbf{y}) \\ &\quad = \lambda \int d\mathbf{w} d\mathbf{w}' v(\mathbf{x} - \mathbf{w}) g_\omega^{[k+1,N]}(\mathbf{x} - \mathbf{w}') W_{-\sigma;\sigma}^{(1;2)(k)}(\mathbf{w}; \mathbf{w}', \mathbf{y}) \end{aligned} \tag{58}$$

so that, from the bound (44), $\|W_{-\sigma;\sigma}^{(1;2)(k)}\|_k \le C$ given by (44), and by the second of (57), we obtain

$$\begin{aligned} \|W_\sigma^{(0;2)(k)}\| &\le |\lambda| \cdot |v|_\infty \cdot \|W_{-\sigma;\sigma}^{(1;2)(k)}\|_k \cdot \sum_{j=k}^N |g_\omega^{(j)}|_1 \\ &\le \frac{c_1}{1-\gamma^{-1}} C |v|_\infty |\lambda| \gamma^{-k} \le \frac{1-\gamma^{-1}}{4c_1} C_1 |\lambda| \gamma^{-k} \end{aligned} \tag{59}$$

which proves the first of (50), since $\frac{1-\gamma^{-1}}{c_1} < 1$ (C_1 is chosen so large to have such a factor because of later usage). Let us consider now $W_{\sigma';\sigma}^{(1;2)(k)}$, which from (53) can be rewritten as in Fig. 5

1. The graph (a) in Fig. 5 is given by:

$$\begin{aligned} & W_{(a)\sigma';\sigma}^{(1;2)(k)}(\mathbf{z}; \mathbf{x}, \mathbf{y}) \\ &\quad \stackrel{def}{=} \lambda \int d\mathbf{w} d\mathbf{u} v(\mathbf{x} - \mathbf{w}) g_\omega^{[k+1,N]}(\mathbf{x} - \mathbf{u}) W_{\sigma',-\sigma;\sigma}^{(2;2)(k)}(\mathbf{z}, \mathbf{w}; \mathbf{u}, \mathbf{y}). \end{aligned} \tag{60}$$

From the bound (44), $\|W_{\sigma',-\sigma;\sigma}^{(2;2)(k)}\|_k \le C^2 \gamma^{-k}$, we obtain

$$\|W_{(a);\sigma';\sigma}^{(1;2)(k)}\|_k \le |\lambda| \cdot |v|_\infty \cdot \|W_{\sigma',-\sigma;\sigma}^{(2;2)(k)}\|_k \cdot \sum_{j=k}^N |g_\omega^{(j)}|_1 \le \frac{C_1}{4} |\lambda| \gamma^{-2k}. \tag{61}$$

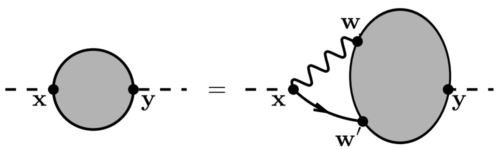


Fig. 4 Topological identity for $W^{(0;2)(k)}$

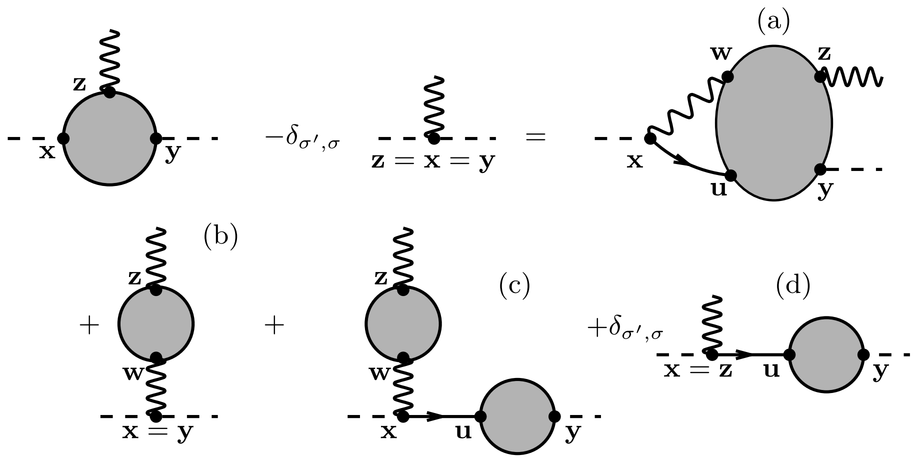


Fig. 5 Topological identity for $W^{(1;2)(k)}$

2. The graph (d) is given by

$$W^{(1;2)(k)}_{(d)\sigma';\sigma}(\mathbf{z};\mathbf{x},\mathbf{y}) \stackrel{def}{=} \delta_{\sigma,\sigma'}\delta(\mathbf{x}-\mathbf{z})\int d\mathbf{u} g^{[k+1,N]}_{\omega}(\mathbf{x}-\mathbf{u})W^{(0;2)(k)}_{\sigma}(\mathbf{u},\mathbf{y}) \tag{62}$$

and using (59) we get

$$\begin{aligned}\|W^{(1;2)(k)}_{(d)\sigma';\sigma}\|_k &\le \delta_{\sigma,\sigma'}\cdot\|W^{(0;2)(k)}_{\sigma}\|_k\cdot\sum_{j=k}^{N}|g^{(j)}_{\omega}|_1\\ &\le \|W^{(0;2)(k)}_{\sigma}\|_k\cdot\frac{c_1}{1-\gamma^{-1}}\gamma^{-k}\le\frac{C_1}{4}|\lambda|\gamma^{-2k}.\end{aligned} \tag{63}$$

In order to obtain an improved bound also for the graphs (b) and (c) of Fig. 5, we need to further expand $W^{(2;0)(k)}_{\sigma';-\sigma}$. Using (56), we find

$$\begin{aligned}&W^{(2;0)(k)}_{\sigma',-\sigma}(\mathbf{z},\mathbf{w})\\ &\quad=\int d\mathbf{u}'d\mathbf{u} g^{[k+1,N]}_{\omega}(\mathbf{w}-\mathbf{u})g^{[k+1,N]}_{\omega}(\mathbf{w}-\mathbf{u}')W^{(1;2)(k)}_{\sigma';-\sigma}(\mathbf{z};\mathbf{u}',\mathbf{u})\end{aligned} \tag{64}$$

and then, replacing the expansion for $W^{(1;2)(k)}_{\sigma';-\sigma}(\mathbf{z};\mathbf{u}',\mathbf{u})$ in the graph (64) we find for (b) what is depicted in Fig. 6.

3. We now consider (b1) of Fig. 6.

$$\begin{aligned}&W^{(1;2)(k)}_{(b1)\sigma';\sigma}(\mathbf{z};\mathbf{x},\mathbf{y})\\ &\quad\stackrel{def}{=}\lambda\delta(\mathbf{x}-\mathbf{y})\int d\mathbf{w}d\mathbf{u}'d\mathbf{z}'v(\mathbf{x}-\mathbf{w})v(\mathbf{u}'-\mathbf{z}')\\ &\qquad\times\int d\mathbf{u}d\mathbf{w}'g^{[k+1,N]}_{\omega}(\mathbf{w}-\mathbf{u})g^{[k+1,N]}_{\omega}(\mathbf{w}-\mathbf{u}')g^{[k+1,N]}_{\omega}(\mathbf{u}'-\mathbf{w}')\\ &\qquad\times W^{(2;2)(k)}_{\sigma',\sigma;-\sigma}(\mathbf{z},\mathbf{z}';\mathbf{w}',\mathbf{u}).\end{aligned} \tag{65}$$

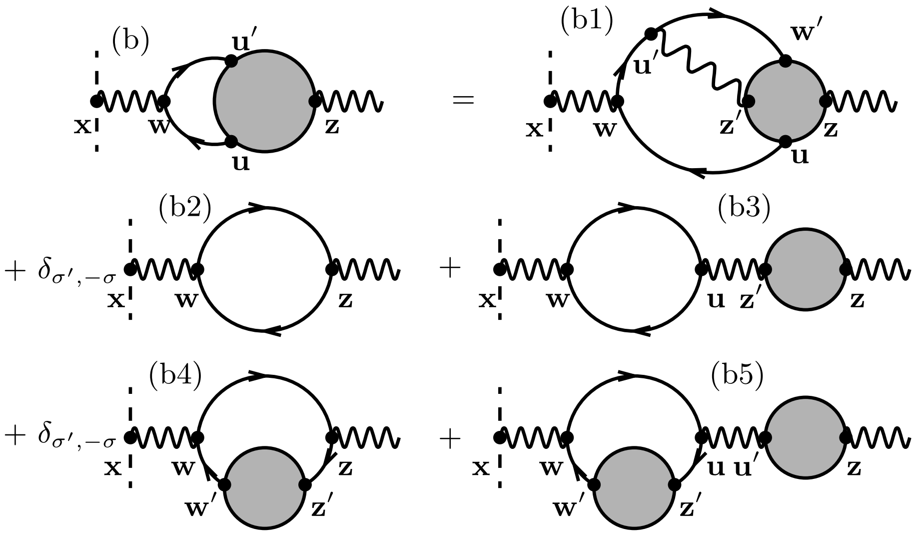


Fig. 6 Graphical representation of graph (b) in Fig. 5

In order to obtain bound uniform in $N-k$, it is convenient to decompose the three propagators $g_\omega g_\omega g_\omega$ into scales, $\sum_{j,i,i'=k}^{N} g_\omega^{(j)} g_\omega^{(i)} g_\omega^{(i')}$ and then, for any realization of j,i,i', to take the $|\cdot|_1$ norm on the two propagator on the higher scales, and the $|\cdot|_\infty$ norm on the propagator with the lowest one. In this way we obtain:

$$\|W^{(1;2)(k)}_{(b1)\sigma';\sigma}\| \leq |\lambda|\cdot|v|_\infty\cdot|v|_1\cdot\|W^{(2;2)(k)}_{\sigma',\sigma,-\sigma}\|_k$$
$$\times\, 3!\sum_{j=k}^{N}\sum_{i=k}^{j}\sum_{i'=k}^{i}|g_\omega^{(j)}|_1|g_\omega^{(i)}|_1|g_\omega^{(i')}|_\infty \leq \frac{C_1}{20}|\lambda|\gamma^{-2k} \tag{66}$$

where, in the last inequality, we have taken $|\lambda|$ small enough, and we have used that $\sum_{j=k}^{N}\gamma^{-j}(j-k)\leq C\sum_{j=k}^{N}\gamma^{-j}\gamma^{(j-k)/2}\leq C'\gamma^{-k}$.

4. The expression for (b2) is:

$$W^{(1;2)(k)}_{(b2)\sigma';\sigma}(\mathbf{z};\mathbf{x},\mathbf{y})$$
$$\stackrel{def}{=}\lambda\delta_{\sigma',-\sigma}\delta(\mathbf{x}-\mathbf{y})\int d\mathbf{w}v(\mathbf{x}-\mathbf{w})[g^{[k+1,N]}_{-\omega}(\mathbf{w}-\mathbf{z})]^2. \tag{67}$$

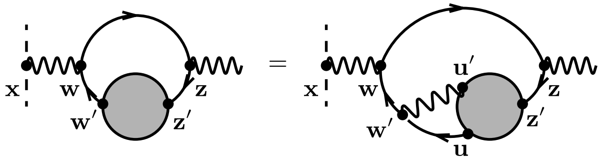
For $\mathbf{k}^*=(-k_0,k)$, it holds $\hat{g}^{[k+1,N]}_\omega(\mathbf{k})=-i\omega\hat{g}^{[k+1,N]}_\omega(\mathbf{k}^*)$ hence

$$\int d\mathbf{u}[g^{[k+1,N]}_{-\omega}(\mathbf{u})]^2=0. \tag{68}$$

Since

$$v(\mathbf{x}-\mathbf{w})=v(\mathbf{x}-\mathbf{z})+\sum_{j=0,1}(z_j-w_j)\int_0^1 d\tau\;(\partial_j v)(\mathbf{x}-\mathbf{z}+\tau(\mathbf{z}-\mathbf{w})) \tag{69}$$

Fig. 7 Equivalent expressions for (b4)



we can write

$$
\begin{aligned}
&W^{(1;2)(k)}_{(b2)\sigma';\sigma}(\mathbf{z};\mathbf{x},\mathbf{y})\\
&\quad=\lambda\delta_{\sigma',-\sigma}\delta(\mathbf{x}-\mathbf{y})v(\mathbf{x}-\mathbf{z})\int d\mathbf{w}[g^{[k+1,N]}_{-\omega}(\mathbf{w})]^2+\lambda\delta_{\sigma',-\sigma}\delta(\mathbf{x}-\mathbf{y})\\
&\qquad\times\sum_{j=0,1}\int_0^1 d\tau\int d\mathbf{w}(\partial_j v)(\mathbf{x}-\mathbf{z}+\tau(\mathbf{z}-\mathbf{w}))(z_j-w_j)g^{[k+1,N]}_{\omega}(\mathbf{w}-\mathbf{z})
\end{aligned}
$$

and the first addend is vanishing because of (68). Hence, using the third of (57),

$$
\begin{aligned}
&\|W^{(1;2)(k)}_{(b2)\sigma';\sigma}\|_k\\
&\quad\le|\lambda|\sum_{j=0,1}\int_0^1 d\tau\int d\mathbf{w}d\mathbf{x}|(\partial_j v)(\mathbf{x}-\mathbf{z}-\tau\mathbf{w})w_j[g^{[k+1,N]}_{-\omega}(\mathbf{w})]^2|\\
&\quad\le 4|\lambda|\int d\mathbf{x}|(\partial_j v)(\mathbf{x})|\sum_{i=k}^{N}\sum_{j=k}^{i}|g^{(j)}_{-\omega}|_\infty\\
&\qquad\times\int d\mathbf{w}|w_j||g^{(i)}_{\omega}(\mathbf{w})|\le|\lambda|\frac{C_1}{20}\gamma^{-k}
\end{aligned}
\tag{70}
$$

5. The expression for (b3) is:

$$
\begin{aligned}
&W^{(1;2)(k)}_{(b2)\sigma';\sigma}(\mathbf{z};\mathbf{x},\mathbf{y})\\
&\quad\overset{def}{=}\lambda\delta_{\sigma',-\sigma}\delta(\mathbf{x}-\mathbf{y})\int d\mathbf{w}v(\mathbf{x}-\mathbf{w})\\
&\qquad\times\lambda\int d\mathbf{z}'[g^{[k+1,N]}_{-\omega}(\mathbf{w}-\mathbf{z}')]^2 v(\mathbf{u}-\mathbf{z}')W^{(2;0)(k)}_{\sigma,\sigma'}(\mathbf{z}',\mathbf{z}).
\end{aligned}
\tag{71}
$$

The improved bound for (b3) is obtained in the same way as for (b1).

$$
\|W^{(1;2)(k)}_{(b3)\sigma';\sigma}\|_k\le\frac{C_1}{20}|\lambda|\gamma^{-k}.
\tag{72}
$$

6. It is convenient to further expand (b4) using the identity (58), which, in the case at hand, is depicted in Fig. 7.

   Thereby, explicit expression for (b4) is

$$
\begin{aligned}
&W^{(1;2)(k)}_{(b4)\sigma';\sigma}(\mathbf{z};\mathbf{x},\mathbf{y})\\
&\quad\overset{def}{=}\delta_{\sigma',-\sigma}\lambda^2\int d\mathbf{z}'d\mathbf{w}v(\mathbf{x}-\mathbf{w})g_\omega(\mathbf{w}-\mathbf{z})
\end{aligned}
$$

$$\times \int d\mathbf{w}'d\mathbf{u}'d\mathbf{u} g_\omega^{[k+1,N]}(\mathbf{w}-\mathbf{w}')g_\omega^{[k+1,N]}(\mathbf{w}'-\mathbf{u})v(\mathbf{w}'-\mathbf{u}')$$
$$\times W^{(1;2)(k)}_{-\sigma;\sigma}(\mathbf{u}';\mathbf{u},\mathbf{z}')g_\omega^{[k+1,N]}(\mathbf{z}'-\mathbf{z}). \tag{73}$$

As in the previous cases, it is convenient first to decompose the propagators $g_\omega(\mathbf{w}-\mathbf{z})g_\omega(\mathbf{w}-\mathbf{w}')g_\omega(\mathbf{w}'-\mathbf{u})$ into scales, $\sum_{j,i,i'=k}^N g_\omega^{(j)}g_\omega^{(i)}g_\omega^{(i')}$ and then, for any realization of j,i,i', to bound with $|\cdot|_1$ norm the two propagators on highest scale, and with $|\cdot|_\infty$ norm the one on lower scale. Finally, for $|\lambda|$ small enough, we have:

$$\|W^{(1;2)(k)}_{(b4);\omega';\omega}\|_k \le \delta_{\sigma',-\sigma}|\lambda|^2\cdot|v|_1\cdot|v|_\infty\cdot\|W^{(1;2)(k)}_{-\sigma;\sigma}\|_k\cdot|g_\omega|_1$$
$$\times 3!\sum_{j=k}^N\sum_{i=k}^j\sum_{i'=k}^i |g_\omega^{(j)}|_1\,|g_\omega^{(i)}|_1\,|g_\omega^{(i')}|_\infty \le \frac{C_1}{20}|\lambda|\gamma^{-2k}. \tag{74}$$

7. Similar arguments can be used to bound also the graph $(b5)$.

Finally, it is also clear that a bound for (c) of Fig. 5 can be found along the same lines discussed for (b) of the same figure. We have so proved, therefore

$$\|W^{(1;2)(k)}_{\sigma';\sigma}\|_k \le \frac{C_1}{C_2}\gamma^{-k} \tag{75}$$

where, for later purposes, C_1 is chosen large enough so that in (75) $C_2 = 1+2|v|_\infty(1+|g|_1\cdot\|W^{(0;2)(k)}_\sigma\|_k)$. Clearly (75) implies the second of (50).

Finally from (53) we obtain the identity in Fig. 8. Therefore the bound for the sum of the graphs (a), (b), (d), and (e) is

$$|\lambda|\cdot|v|_1\cdot\|W^{(1;2)(k)}_{\sigma;\sigma'}-\delta_{\sigma,\sigma'}\delta_2\|_k(1+|g|_1\cdot\|W^{(0;2)(k)}_\sigma\|_k) \le \frac{C_1}{2}\gamma^{-k}. \tag{76}$$

Indeed, the last inequality follows from the just proved, improved bound $\|W^{(1;2)(k)}_{\sigma;\sigma'}-\delta_{\sigma,\sigma'}\delta_2\|_k \le \frac{C_1}{C_2}|\lambda|\gamma^{-k}$. Finally, the graph (c) is

$$W^{(0;4)}_{(a),\omega;\sigma,\sigma'}(\mathbf{x},\mathbf{y},\mathbf{x}',\mathbf{y}')$$
$$\stackrel{def}{=}\lambda\int d\mathbf{w}d\mathbf{u}v(\mathbf{x}-\mathbf{w})g_\omega^{[k+1,N]}(\mathbf{x}-\mathbf{u})W^{(1;4)}_{-\sigma;\sigma,\sigma'}(\mathbf{w};\mathbf{u},\mathbf{y},\mathbf{x}',\mathbf{y}'). \tag{77}$$

Using (44), $\|W^{(1;4)}_{-\sigma;\sigma,\sigma'}\| \le C|\lambda|\gamma^{-k}$ and

$$\|W^{(0;4)}_{(a),\omega;\sigma,\sigma'}\|_k \le |\lambda|\cdot|v|_\infty\cdot|g_\omega|_1\cdot\|W^{(1;4)}_{-\sigma;\sigma,\sigma'}\|_k \le \frac{C_1}{2}|\lambda|\gamma^{-2k}. \tag{78}$$

From this the third of (50) follows and the theorem is proved. □

4 Schwinger Functions

The multiscale integration of (4), when $\varphi \ne 0$, is obtained by a slight modification of the one presented in Sect. 2. In particular $\mathcal{V}^{(k)}(\psi^{(\le k)},\phi,J)$ is given by an expression similar to

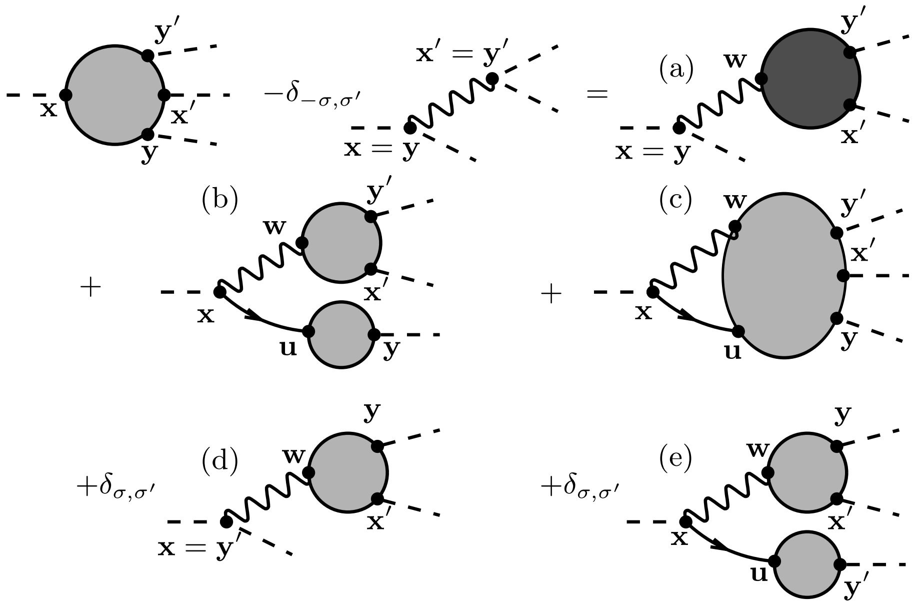


Fig. 8 Graphical representation of $W^{(0;4)(k)}$. The *dark bubble* represents $W^{(1;2)(k)}_{\sigma;\sigma'} - \delta_{\sigma,\sigma'}\delta_2$

(13), sum of monomials in $\psi^{(\le k)}$, J and ϕ. We define $\mathcal{L} = 0$ on the kernels of the monomials containing at least a ϕ except when the monomial is $\varphi^{+}_{\mathbf{x},\omega,\sigma}\psi^{-(\le k+1)}_{\mathbf{y},\omega,\sigma}$ or $\psi^{+(\le k+1)}_{\mathbf{y},\omega,\sigma}\varphi^{-}_{\mathbf{x},\omega,\sigma}$; in such a case the kernel is $\hat{g}_\omega(\mathbf{k})\hat{W}^{(0;2)(k)}_\sigma(\mathbf{k})$ and we define, for $0 \le k \le N$,

$$\mathcal{L}[\hat{g}_\omega(\mathbf{k})\hat{W}^{(0;2)(k)}_\sigma(\mathbf{k})] \stackrel{def}{=} \hat{g}_\omega(\mathbf{k})\hat{W}^{(0;2)(k)}_\sigma(\mathbf{k}) \tag{79}$$

while, for $k < 0$, $\mathcal{L} \equiv 0$. Correspondingly, for $k > 0$ we define

$$\gamma^{-k}\tilde{\nu}_{k,\omega,\sigma}(\mathbf{x},\mathbf{y}) \stackrel{def}{=} \int d\mathbf{z} g_\omega(\mathbf{x}-\mathbf{z}) W^{(0;2)(k)}_\sigma(\mathbf{z},\mathbf{y}) \tag{80}$$

and using (59) we obtain $\|\tilde{\nu}_{k,\omega,\sigma}\|_k \le C_1|\lambda|\gamma^{-k}$; while for $k < 0$ we set $\tilde{\nu}_{k,\omega,\sigma}(\mathbf{x},\mathbf{y}) \equiv 0$, because of the fact that $\hat{W}^{0;2(k)}_\sigma(0) = 0$ by symmetries, and then there is an automatic dimensional gain:

$$\hat{g}_\omega(\mathbf{k})\hat{W}^{(0;2)(k)}_\sigma(\mathbf{k}) = \hat{g}_\omega(\mathbf{k})[\hat{W}^{(0;2)(k)}_\sigma(\mathbf{k}) - \hat{W}^{(0;2)(k)}_\sigma(0)]. \tag{81}$$

Let $\tilde{\varepsilon}_k$ be larger than ε_k and $\max_{\omega,\sigma}\max_{h:k\le h\le N}\|\tilde{\nu}_{k,\omega,\sigma}\|_k$. The 2-points Schwinger function is given by

$$S_{N;\omega;\sigma}(\mathbf{x},\mathbf{y}) = \sum_{h\le N} g^{(h)}_\omega(\mathbf{x}-\mathbf{y}) + \sum_{n=0}^{\infty}\sum_{j\le N}\sum_{\tau\in\bar{\mathcal{T}}^{2,0}_{j,n}}\sum_{\substack{\mathbf{P}\in\mathcal{P}\\ |P_{v_0}|=2}} S_\tau(\mathbf{x},\mathbf{y}), \tag{82}$$

where $\bar{\mathcal{T}}^{n^\varphi,n^J}_{h,n}$ is the set of trees with n endpoints, n^φ special endpoints of type φ, n^J end-points of type J and first vertex scale j; n^J_v, n^ϕ_v are the number of fields of type J, ϕ associated to end-points following v. If h is the first nontrivial vertex u of τ, and h_1 and h_2 are the scale of the two endpoints of type φ, we have

$$|S_\tau(\mathbf{x},\mathbf{y})| \le \tilde{C}_q(c\tilde{\varepsilon}_h)^{n-2}\gamma^{j-h_1-h_2}\prod_{v \text{ not e.p.}}\gamma^{-(\frac{|P_v|}{2}-2+z_v)}$$
$$\times\frac{\gamma^{2h}}{1+[\gamma^h|\mathbf{x}-\mathbf{y}|]^{\frac{q}{2}}}. \qquad (83)$$

Indeed, (83) is the same of (43), for $|P_{v_0}|=2$, $n^J_{v_0}=0$, times some factors more.

1. The presence, with respect to the graphical expansion of the kernels, of two external propagators, $g^{(h_1)}_\omega$ and $g^{(h_2)}_\omega$, causes the factor $\gamma^{-h_1-h_2}$.
2. Before performing the bounds as for the kernels, it is possible to extract from the bound on the propagator (11) a factor $b_h=(1+(\gamma^h|\mathbf{x}-\mathbf{y}|)^{\frac{q}{2}})^{-1}$: the product of b_h for each of the propagators of the graph that are not involved into the Gram determinant (18) can be bounded with the factor $[1+[\gamma^h|\mathbf{x}-\mathbf{y}|]^{\frac{q}{2}}]^{-1}$ in (83) at the price of a constant C^n.
3. The bounds for the kernels can be straightforwardly modified also for obtaining the factor γ^{2h}: it is the effect of the missed integration in the variable $\mathbf{x}-\mathbf{y}$, that causes the replacement of $|\cdot|_1$-norm with the $|\cdot|_\infty$-norm of a propagator g_ω; this occurs in correspondence of v, the vertex with highest scale, h, in which the two special endpoints of type φ are connected.

It is convenient to call $|P_v|=n^\psi_v+n^\varphi_v$. We have that z_v is the same of (43), with a further case in which it is not zero: if $h_v<0$ and $n^\psi_v=n^\varphi_v=1$, then $z_v=1$. This is because the automatic dimensional gain depicted in (81).

Along the tree τ, we consider three paths: $\mathcal{C}_1$ and $\mathcal{C}_2$, connecting the endpoint of type φ on scale h_1 and the one on scale h_2 respectively with v_0; and $\mathcal{C}$ connecting u with v_0. For $j=1,2$, we find $\gamma^{-h_j}=\gamma^{-j}\prod_{v\in\mathcal{C}_j}\gamma^{-1}$ and $\gamma^{-j}=\gamma^{-h}\prod_{v\in\mathcal{C}}\gamma$. These identities, replaced in (83), gives:

$$|S_\tau(\mathbf{x},\mathbf{y})| \le \tilde{C}_q(c\tilde{\varepsilon}_h)^{n-2}\frac{\gamma^h}{1+[\gamma^h|\mathbf{x}-\mathbf{y}|]^{\frac{q}{2}}}$$
$$\times\left(\prod_{v \text{ not e.p.}}^{v\notin\mathcal{C}}\gamma^{-(\frac{n^\psi_v}{2}+\frac{3n^\varphi_v}{2}-2+z_v)}\right)\prod_{v\in\mathcal{C}}\gamma^{-\frac{n^\psi_v}{2}} \qquad (84)$$

and $\frac{n^\psi_v}{2}+\frac{3n^\varphi_v}{2}-2+z_v>0$, as well as $\frac{n^\psi_v}{2}>0$ for $v\in\mathcal{C}$: we can perform the summation on the trees, keeping fixed the scale h.

$$|S_{N;\omega;\sigma}(\mathbf{x},\mathbf{y})-g^{(\le N)}_\omega(\mathbf{x},\mathbf{y})| \le C|\lambda|\sum_{h\le N}\frac{\gamma^h}{1+[\gamma^h|\mathbf{x}-\mathbf{y}|]^{\frac{q}{2}}} \le C|\lambda|\frac{1}{|\mathbf{x}-\mathbf{y}|}. \qquad (85)$$

Finally, we want to study the difference $S_{N;\omega;\sigma}(\mathbf{x},\mathbf{y})-S_{\omega;\sigma}(\mathbf{x},\mathbf{y})$ for $\mathbf{x}-\mathbf{y}\ne 0$.

$$S_{N;\omega;\sigma}(\mathbf{x},\mathbf{y})-S_{\omega;\sigma}(\mathbf{x},\mathbf{y})$$
$$=\sum_{h\le N}g^{(h)}_\omega(\mathbf{x}-\mathbf{y})+\sum_{n=0}^{\infty}\sum_{j=-\infty}^{+\infty}\sum_{\tau\in\mathcal{T}^{2,0}_{j,n}}\sum_{\substack{\mathbf{P}\in\mathcal{P}\\|P_{v_0}|=2}}D_\tau(\mathbf{x},\mathbf{y}). \qquad (86)$$

In such a tree expansion, $D_\tau(\mathbf{x},\mathbf{y})$ is not zero only in two cases: either τ has at least one vertex v^* on scale $h^* > N$; or τ has vertices scales $\leq N$, but has an endpoint which, in turn, has tree expansion with at least one vertex v^* on scale $h^* > N$. If τ is of the former type, fixed ϑ, we have

$$|D_\tau(\mathbf{x},\mathbf{y})| \leq \tilde{C}_q (c\tilde{\varepsilon}_h)^{n-2} \frac{\gamma^h}{1+[\gamma^h|\mathbf{x}-\mathbf{y}|]^{\frac{q}{2}}} \times \gamma^{-\vartheta(h^*-h)} \left(\prod_{v \text{ not e.p.}}^{v\notin C} \gamma^{-(\frac{n_v^\psi}{2}+\frac{3n_v^\varphi}{2}-2+z_v-\vartheta)} \right) \prod_{v\in C} \gamma^{-\frac{n_v^\psi}{2}}. \tag{87}$$

It τ is of the latter type, we still have the above bound, by induction on the subtrees in which the endpoints can be expanded: indeed, in the analysis of the previous section it is clear that if the fermion propagator is constrained to be on scale $> N$, bounds (50) are still true, with a more factor $\gamma^{-\vartheta(h^*-k)}$, which, together to a factor $\gamma^{-\vartheta(k-h)}$ gives the wanted $\gamma^{-\vartheta(h^*-h)}$.

For $\vartheta > 0$ and but small enough, we still have $\frac{n_v^\psi}{2} + \frac{3n_v^\varphi}{2} - 2 + z_v - \vartheta > 0$. This means that we can perform the summation on the trees, keeping fixed the scale k. As $\gamma^{-\vartheta(h^*-h)} \leq \gamma^{-\vartheta(N-h)}$,

$$\begin{aligned} |S_{N;\omega;\sigma}(\mathbf{x},\mathbf{y}) - S_{\omega;\sigma}(\mathbf{x},\mathbf{y})| &\leq C\gamma^{-\vartheta N} \sum_{h=-\infty}^{+\infty} \frac{\gamma^{(1+\vartheta)h}}{1+[\gamma^h|\mathbf{x}-\mathbf{y}|]^{\frac{q}{2}}} \\ &\leq C\frac{1}{\gamma^{\vartheta N}|\mathbf{x}-\mathbf{y}|^{1+\vartheta}}. \end{aligned} \tag{88}$$

5 Ward Identities

Let us consider the 2-point Schwinger function with one density insertion:

$$\hat{G}_{N;\omega,\sigma';\sigma}(\mathbf{p};\mathbf{k}) = \frac{\partial^3 \mathcal{W}}{\partial \hat{J}_{\mathbf{p},\omega,\sigma'} \partial \hat{\psi}^+_{\mathbf{k}+\mathbf{p},\omega,\sigma} \partial \hat{\psi}^-_{\mathbf{k},\omega,\sigma}}(0,0). \tag{89}$$

In the generating functional (4), we perform the phase-chiral transformation

$$\hat{\psi}^\varepsilon_{\mathbf{k},\omega,\sigma} \to \hat{\psi}^\varepsilon_{\mathbf{k},\omega,\sigma} + \varepsilon \int \frac{d\mathbf{p}}{(2\pi)^2} \hat{\alpha}_{\mathbf{p},\omega,\sigma} \hat{\psi}^\varepsilon_{\mathbf{k}+\varepsilon\mathbf{p},\omega,\sigma} \tag{90}$$

and obtain the identities:

$$\begin{aligned} &D_\omega(\mathbf{p})\hat{G}_{N;\omega,\sigma';\sigma}(\mathbf{p};\mathbf{k}) \\ &\quad = \delta_{\sigma,\sigma'}[\hat{S}_{N;\omega,\sigma}(\mathbf{k}) - \hat{S}_{N;\omega,\sigma}(\mathbf{k}+\mathbf{p})] + \Delta_{\omega,\sigma';\sigma}(\mathbf{p};\mathbf{k}) \end{aligned} \tag{91}$$

where $\Delta_{\omega;\sigma',\sigma}(\mathbf{p};\mathbf{k})$ is a correction term caused by the presence of the cutoff:

$$\Delta_{\omega;\sigma',\sigma}(\mathbf{p};\mathbf{k}) = \int \frac{d\mathbf{q}}{(2\pi)^2} C_{N;\omega}(\mathbf{q}+\mathbf{p},\mathbf{q}) \langle \hat{\psi}^+_{\mathbf{p}+\mathbf{q},\omega,\sigma'} \hat{\psi}^-_{\mathbf{p},\omega,\sigma'} \hat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega,\sigma} \hat{\psi}^+_{\mathbf{k},\omega,\sigma} \rangle$$

for

$$C_{N;\omega}(\mathbf{k}+\mathbf{p},\mathbf{k}) \stackrel{def}{=} D_\omega(\mathbf{k}+\mathbf{p})[1-\chi_N^{-1}(\mathbf{k}+\mathbf{p})] - D_\omega(\mathbf{k})[1-\chi_N^{-1}(\mathbf{k})].$$

The rest $\Delta_{\omega;\sigma',\sigma}(\mathbf{p};\mathbf{k})$ does not vanish in the limit of removed cutoff, but rather it causes the *anomaly* of the Ward Identities.

Theorem 3 *There exists $\varepsilon_0 > 0$ such that, for $|\lambda| \le \varepsilon_0$ and in the limit of removed cutoff,*

$$\hat{G}_{\omega,\sigma';\sigma}(\mathbf{p};\mathbf{k}) = \frac{a(\mathbf{p}) + \sigma\sigma'\bar{a}(\mathbf{p})}{2}[\hat{S}_{\omega,\sigma}(\mathbf{k}) - \hat{S}_{\omega,\sigma}(\mathbf{k})] \tag{92}$$

for

$$a(\mathbf{p}) = \frac{1}{D_\omega(\mathbf{p}) - \frac{\lambda}{2\pi}\hat{v}(\mathbf{p})D_{-\omega}(\mathbf{p})}, \qquad \bar{a}_N(\mathbf{p}) = \frac{1}{D_\omega(\mathbf{p}) + \frac{\lambda}{2\pi}\hat{v}(\mathbf{p})D_{-\omega}(\mathbf{p})}.$$

The proof is a consequence of the two following lemmas.

Lemma 4 *For $|\lambda|$ small enough and $\mathbf{p}, \mathbf{k}, \mathbf{p} - \mathbf{k} \neq 0$, the limit of removed cutoff of $\hat{G}_{\omega,\sigma';\sigma}(\mathbf{p};\mathbf{k})$ exist and is finite.*

Proof We can write

$$\hat{G}_{\sigma';\sigma}(\mathbf{p};\mathbf{k}) = \sum_{n=0}^{\infty}\sum_{j\le N}\sum_{\tau\in\mathcal{T}^{2,1}_{j,n}}\sum_{\substack{\mathbf{P}\in\mathcal{P}\\ |P_{v_0}|=2}} \hat{G}_\tau(\mathbf{p};\mathbf{k}), \tag{93}$$

with an obvious definition of $\hat{G}_\tau(\mathbf{p},\mathbf{k})$. We define $h_{\mathbf{p}} = \min\{j : f_j(\mathbf{p}) \neq 0\}$ and suppose that $\mathbf{p}$, $\mathbf{k}$, $\mathbf{p}-\mathbf{k}$ are all different from 0. It follows that, given τ, if h_- and h_+ are the scale indices of the ψ fields belonging to the endpoints associated with φ^+ and φ^-, while h_J denotes the scale of the endpoint of type J, $\hat{G}_\tau(\mathbf{p};\mathbf{k})$ can be different from 0 only if $h_- = h_{\mathbf{k}}, h_{\mathbf{k}}+1$, $h_+ = h_{\mathbf{k-p}}, h_{\mathbf{k-p}}+1$ and $h_J \ge h_{\mathbf{p}} - \log_\gamma 2$. Moreover, if $\mathcal{T}^{\mathbf{p},\mathbf{k}}_{j_0,n}$ denotes the set of trees satisfying the previous conditions and $\tau \in \mathcal{T}^{\mathbf{p},\mathbf{k}}_{j_0,n}$, $|\hat{G}_\tau(\mathbf{p};\mathbf{k})|$ can be bounded by $\int d\mathbf{z}d\mathbf{x}|G_\tau(\mathbf{z};\mathbf{x},\mathbf{y})|$. We get

$$|\hat{G}^{(1;2)}_{\sigma';\sigma}(\mathbf{p};\mathbf{k})| \le C\gamma^{-h_{\mathbf{k}}}\gamma^{-h_{\mathbf{k-p}}}$$
$$\times \sum_{n=0}^{\infty}\sum_{j\le N}\sum_{\tau\in\mathcal{T}^{\mathbf{p},\mathbf{k}}_{j_0,n}}\sum_{\substack{\mathbf{P}\in\mathcal{P}\\ |P_{v_0}|=2}} (C|\lambda|)^n \prod_{\text{v not e.p}} \gamma^{-d_v}, \tag{94}$$

where $d_v = \frac{|P_v|}{2} - 2 + z_v + n^\phi_v$.

Given $\tau \in \mathcal{T}^{\mathbf{p},\mathbf{k}}_{j_0,n}$, let v^*_0 the higher vertex preceding all three special endpoints and $v^*_1 \ge v^*_0$ the higher vertex preceding either the two endpoints of type φ or one endpoint of type φ and the endpoint of type J. We have $d_v > 0$, except for a finite number of vertices belonging to the path $\mathcal{C}^*$ connecting v^*_1 with v^*_0, where $d_v = 0$:

(a) the vertices with $|P_v| = 4$ and $n^J_v = 0$; since there is a momentum $\mathbf{k}$ flowing inside the corresponding cluster and $\mathbf{k}-\mathbf{p}$ flowing outside, by conservation of the momenta the scale label of both of the other ψ fields—and hence also the scale label of such vertices—cannot be less than $\log_\gamma(|\mathbf{p}|/2)$;
(b) the vertices with $|P_v| = 2$ and $n^J_v = 1$; with a momentum $\mathbf{p}$ flowing inside the cluster and either a momentum $\mathbf{k}$ flowing inside or $\mathbf{k}-\mathbf{p}$ flowing outside, the scale label of such vertices cannot be less than $\min\{h_+, h_-\} - 1$.

Accordingly, the number of the vertices depicted in the above list is not larger than $\min\{|h_{\mathbf{k}} - h_{\mathbf{p}}|, |h_{\mathbf{k-p}} - h_{\mathbf{p}}|\} + 2 - \log_\gamma 2$. Thus we can replace in (94) the rough bound:

$$\prod_{\text{v not e.p}} \gamma^{-d_v} \leq C\gamma^{|h_{\mathbf{k}}-h_{\mathbf{p}}|}\gamma^{|h_{\mathbf{k-p}}-h_{\mathbf{p}}|} \prod_{\text{v not e.p}} \gamma^{-d_v - r_v}$$

with $r_v = 1$ for $v : d_v = 0$ and $r_v = 0$ otherwise. Finally, we can perform the sums over the scale and P_v labels of τ, obtaining:

$$|\hat{G}_{\sigma';\sigma'}(\mathbf{p};\mathbf{k})| \leq C\gamma^{-h_{\mathbf{k}}}\gamma^{-h_{\mathbf{k-p}}}\gamma^{|h_{\mathbf{k}}-h_{\mathbf{p}}|}\gamma^{|h_{\mathbf{k-p}}-h_{\mathbf{p}}|}. \tag{95}$$

This completes the proof. □

Lemma 5 *There exist a finite ν_N such that it is possible to decompose*

$$\begin{aligned} &\Delta^{(1;2)}_{\omega;\sigma',\sigma}(\mathbf{p};\mathbf{k}) - \nu_N \hat{v}(\mathbf{p}) D_{-\omega}(\mathbf{p}) \hat{G}_{N;\omega;-\sigma',\sigma}(\mathbf{p};\mathbf{k}) \\ &\quad = \sum_{\bar{\omega}} D_{\bar{\omega}}(\mathbf{p}) \hat{R}^{(1;2)}_{N;\bar{\omega},\omega;\sigma',\sigma}(\mathbf{p};\mathbf{k}) \end{aligned} \tag{96}$$

where $\hat{R}^{(1;2)}_{N;\bar{\omega},\omega;\sigma',\sigma}$ is such that, for fixed $\mathbf{k}$ *and* $\mathbf{p}$, *it holds*

$$\lim_{N\to\infty} \hat{R}^{(1;2)}_{N;\bar{\omega},\omega;\sigma',\sigma}(\mathbf{p};\mathbf{k}) = 0. \tag{97}$$

Furthermore, $\lim_{N\to\infty} \nu_N = \frac{\lambda}{2\pi}$.

Proof It is convenient to write the rest $\hat{R}^{(1;2)}_{\bar{\omega},\omega,\sigma';\sigma}$ as

$$\sum_{\omega'} D_{\omega'}(\mathbf{q}) \hat{R}^{(1;2)}_{\bar{\omega},\omega,\sigma';\sigma}(\mathbf{q};\mathbf{k}) = \frac{\partial^3 \mathcal{W}_\Delta}{\partial\hat{\alpha}_{\mathbf{q},\omega,-\sigma'}\partial\hat{\varphi}^+_{\mathbf{k-q},\omega,\sigma}\partial\hat{\varphi}^-_{\mathbf{k},\omega,\sigma}}(0,0) \tag{98}$$

where we have introduced the new generating functional $\mathcal{W}_\Delta(\alpha,\varphi)$ defined such that:

$$\begin{aligned} e^{\mathcal{W}_\Delta(\alpha,\varphi)} &= \int P(d\psi^{\leq N})\, e^{-\mathcal{V}^{(N)}_\Delta(\psi^{(\leq N)},\alpha,\varphi)} \\ &\stackrel{def}{=} \int P(d\psi^{\leq N}) \exp\{-\lambda V(\psi^{(\leq N)}) + [T_0 - \nu_N T_-](\psi^{(\leq N)},\alpha)\} \\ &\quad \times \exp\left\{\sum_{\omega,\sigma}\int d\mathbf{z}(\psi^{(\leq N)+}_{\mathbf{z},\omega,\sigma}\varphi^-_{\mathbf{z},\omega,\sigma} + \varphi^+_{\mathbf{z},\omega,\sigma}\psi^{(\leq N)-}_{\mathbf{z},\omega,\sigma})\right\} \end{aligned} \tag{99}$$

with

$$\begin{aligned} T_0(\psi,\alpha) &= \sum_{\omega,\sigma}\int \frac{d\mathbf{p}d\mathbf{q}}{(2\pi)^4} \bar{\chi}_N(\mathbf{p}) C_{N;\omega}(\mathbf{q}+\mathbf{p},\mathbf{q}) \hat{\alpha}_{\mathbf{p},\omega,\sigma} \hat{\psi}^+_{\mathbf{q+p},\omega,\sigma} \hat{\psi}^-_{\mathbf{q},\omega,\sigma}, \\ T_-(\psi,\alpha) &= \sum_{\omega,\sigma}\int \frac{d\mathbf{p}\,d\mathbf{q}}{(2\pi)^4} \bar{\chi}_N(\mathbf{p}) \hat{v}(\mathbf{p}) D_{-\omega}(\mathbf{p}) \hat{\alpha}_{\mathbf{p},\omega,\sigma} \hat{\psi}^+_{\mathbf{q+p},\omega,-\sigma} \hat{\psi}^-_{\mathbf{q},\omega,-\sigma}. \end{aligned} \tag{100}$$

We remark that the presence of the cutoff function $\bar\chi_N(\mathbf{p}) \stackrel{def}{=} \sum_{j=-N}^N \hat f_j(\mathbf{p})$ is immaterial for (98), since $\mathbf{p}$ is finite and nonzero. But it is essential for the multiscale integration, because it simplify the discussion of the *tadpoles*.

A crucial role in the following analysis is played by the functions

$$\begin{aligned}\hat U^{(i,j)}_\omega(\mathbf{q}+\mathbf{p},\mathbf{q}) &\stackrel{def}{=} \bar\chi_N(\mathbf{p}) C_{N;\omega}(\mathbf{q}+\mathbf{p},\mathbf{q})\hat g^{(i)}_\omega(\mathbf{q}+\mathbf{p})\hat g^{(j)}_\omega(\mathbf{q}),\\ \hat Q^{(N,i)}_\omega(\mathbf{q}+\mathbf{p},\mathbf{q}) &\stackrel{def}{=} \bar\chi_N(\mathbf{p}) C_{N;\omega}(\mathbf{q}+\mathbf{p},\mathbf{q})\hat g^{(N)}_\omega(\mathbf{q}+\mathbf{p})\hat \chi_j(\mathbf{q}).\end{aligned} \tag{101}$$

We remark that $\hat U^{(i,j)}_\omega(\mathbf{p},\mathbf{q}) = \hat U^{(j,i)}_\omega(\mathbf{q},\mathbf{p})$; in particular $\hat U^{(i,j)}_\omega \equiv 0$ if neither j nor i equals N. As proved in [3] (see also Appendix A) it is possible to decompose

$$\begin{aligned}\hat U^{(i,j)}_\omega(\mathbf{q}+\mathbf{p},\mathbf{q}) &= \sum_{\bar\omega} D_{\bar\omega}(\mathbf{p})\hat S^{(i,j)}_{\bar\omega,\omega}(\mathbf{q}+\mathbf{p},\mathbf{q}),\\ \hat Q^{(N,j)}_\omega(\mathbf{q}+\mathbf{p},\mathbf{q}) &= \sum_{\bar\omega} D_{\bar\omega}(\mathbf{p})\hat P^{(N,i)}_{\bar\omega,\omega}(\mathbf{q}+\mathbf{p},\mathbf{q})\end{aligned} \tag{102}$$

for $\hat S^{(i,j)}_{\bar\omega,\omega}$ such that, calling

$$S^{(i,j)}_{\bar\omega,\omega}(\mathbf{z};\mathbf{x},\mathbf{y}) = \int \frac{d\mathbf{p}d\mathbf{q}}{(2\pi)^4} e^{-i\mathbf{p}(\mathbf{x}-\mathbf{z})}e^{-i\mathbf{q}(\mathbf{y}-\mathbf{z})}\hat S^{(i,j)}_{\bar\omega,\omega}(\mathbf{p},\mathbf{q}) \tag{103}$$

and similarly for $P^{(N,j)}_{\bar\omega,\omega}$, for any positive integers p,q there exists a constant $C_{p,q}>1$ such that

$$\begin{aligned}|S^{(N,j)}_{\bar\omega,\omega}(\mathbf{z};\mathbf{x},\mathbf{y})| &\le C_{p,q}\frac{\gamma^N}{1+[\gamma^N|\mathbf{x}-\mathbf{z}|]^p}\frac{\gamma^j}{1+[\gamma^j|\mathbf{y}-\mathbf{z}|]^q},\\ |P^{(N,j)}_{\bar\omega,\omega}(\mathbf{z};\mathbf{x},\mathbf{y})| &\le C_{p,q}\frac{\gamma^N}{1+[\gamma^N|\mathbf{x}-\mathbf{z}|]^p}\frac{\gamma^{2j}}{1+[\gamma^j|\mathbf{y}-\mathbf{z}|]^q}.\end{aligned} \tag{104}$$

The lemma holds if we choose ν_N to be

$$\nu_N \stackrel{def}{=} \sum_{i,j=-\infty}^{N}\int \frac{d\mathbf{p}}{(2\pi)^2}\hat S^{(i,j)}_{-\omega,\omega}(\mathbf{p},\mathbf{p}). \tag{105}$$

As proved in [5], in the limit of removed cutoff of (105) equals $\frac{\lambda}{2\pi}$. Therefore we have to prove that with this choice (97) holds true.

The integration of $\mathcal{W}_\Delta(\alpha,\varphi)$ is done by a multiscale integration similar to the previous one. After the integration of the fields $\psi^{(N)},\dots,\psi^{(k+1)}$ we get the effective potential $\mathcal{V}^{(k)}_\Delta$ such that

$$e^{-\mathcal{V}^{(k)}_\Delta(\psi^{(\le k)},\alpha,\varphi)} = \int P(d\psi^{[k+1,N]})e^{-\mathcal{V}^{(N)}_\Delta(\psi^{(\le N)},\alpha,\varphi)}. \tag{106}$$

In particular, in view of (98), we are interested in the part of $\mathcal{V}^{(k)}_\Delta(\psi^{(\le k)},\alpha,\varphi)$ linear in α, that we call $\mathcal{K}^{(k)}_\Delta(\psi^{(\le k)},\alpha,\varphi)$. We first consider the kernels for $\varphi=0$.

$$\begin{aligned}&\mathcal{K}^{(k)}_\Delta(\psi,\alpha,0)\\ &\quad= \sum_{m\ge1}\sum_{\underline\omega,\underline\sigma}\int d\underline{\mathbf{x}}d\underline{\mathbf{y}}d\underline{\mathbf{z}}\frac{K^{(1;2m)(k)}_{\Delta;\omega,\sigma,\underline\sigma}(\underline{\mathbf{x}};\underline{\mathbf{y}},\underline{\mathbf{z}})}{2m!}\alpha_{\mathbf{x},\omega,\sigma}\prod_{i=1}^m\psi^+_{\mathbf{y},\omega,\sigma_i}\prod_{i=1}^m\psi^-_{\mathbf{z},\omega,\sigma_i}.\end{aligned} \tag{107}$$

As consequence of (102), we decompose

$$\hat{K}^{(1;2m)(k)}_{\Delta;\omega;\sigma,\underline{\sigma}}(\mathbf{p};\underline{\mathbf{k}}) \stackrel{def}{=} \sum_{\bar{\omega}} D_{\bar{\omega}}(\mathbf{p})\hat{W}^{(1;2m)(k)}_{\Delta;\bar{\omega},\omega;\sigma,\underline{\sigma}}(\mathbf{p};\underline{\mathbf{k}}). \tag{108}$$

We prove the following result.

Lemma 6 *For $|\lambda|$ small enough and $\mathbf{p},\mathbf{k},\mathbf{p}+\mathbf{k}\neq 0$, we have that $W^{(1;2m)(k)}_{\Delta;\omega,\sigma,\underline{\sigma}}$ analytic in λ and, for $m\geq 1$,*

$$\|W^{(1;2m)(k)}_{\Delta;\omega,\sigma,\underline{\sigma}}\|_k \leq C|\lambda|\gamma^{-\frac{1}{2}(N-k)}\gamma^{k(1-m)} \tag{109}$$

Proof We integrate as in (33), and the difference with respect to (4) is that the term $\int d\mathbf{x} J_{\mathbf{x},\omega,\sigma}\psi^+_{\mathbf{x},\omega,\sigma}\psi^-_{\mathbf{x},\omega,\sigma}$ is replaced by $T_0(\psi,\alpha)-\nu_N T_-(\psi,\alpha)$. The integration is done exactly as in Sect. 2; we define for $0\leq k\leq N$

$$\mathcal{L}\hat{W}^{(1;2)(k)}_{\Delta;\varepsilon\omega,\omega,\sigma,\underline{\sigma}}(\mathbf{p};\mathbf{k}) = \hat{W}^{(1;2)(k)}_{\Delta;\varepsilon\omega,\omega,\sigma,\underline{\sigma}}(\mathbf{p};\mathbf{k}) \stackrel{def}{=} \hat{\nu}^{\varepsilon}_{k,\omega,\sigma}(\mathbf{p};\mathbf{k}) \tag{110}$$

so that for $k\geq 0$

$$\begin{aligned}\mathcal{L}\mathcal{V}^{(k)}_{\Delta}&(\psi^{(\leq k)},\alpha,0)\\ &= \mathcal{L}\mathcal{V}(\psi^{(\leq k)},0)\\ &\quad+\sum_{\varepsilon=\pm}\int\frac{d\mathbf{k}d\mathbf{p}}{(2\pi)^4}D_{\varepsilon\omega}(\mathbf{p})\hat{\nu}^{\varepsilon}_{k,\omega,\sigma}(\mathbf{p};\mathbf{k})\hat{\alpha}_{\mathbf{p},\omega,\sigma}\hat{\psi}^{(\leq k)+}_{\mathbf{k},\omega,\varepsilon\sigma}\hat{\psi}^{(\leq k)-}_{\mathbf{k}+\mathbf{p},\omega,\varepsilon\sigma}.\end{aligned} \tag{111}$$

$\mathcal{L}\mathcal{V}(\psi^{(\leq k)},0)$ is given by the first two addenda of (38). On the other hand for $k\leq 0$ we define

$$\mathcal{L}\hat{W}^{(1;2)(k)}_{\Delta;\varepsilon\omega,\omega,\sigma,\underline{\sigma}}(\mathbf{p};\mathbf{k}) = \hat{W}^{(1;2m)(k)}_{\Delta;\varepsilon\omega,\omega,\sigma,\underline{\sigma}}(0;0) \stackrel{def}{=} \hat{\nu}^{\varepsilon}_{k,\omega,\sigma} \tag{112}$$

so that we define $\hat{\nu}^+_{k,\omega,\sigma}$ and $\hat{\nu}^-_{k,\omega,\sigma}$ such that

$$\begin{aligned}\mathcal{L}\mathcal{V}^{(k)}_{\Delta} &= \mathcal{L}\mathcal{V}(\psi^{(\leq k)},0)\\ &\quad+\sum_{\varepsilon=\pm}\hat{\nu}^{\varepsilon}_{k,\omega,\sigma}\int\frac{d\mathbf{k}d\mathbf{p}}{(2\pi)^4}D_{\varepsilon\omega}(\mathbf{p})\alpha_{\mathbf{p},\omega,\sigma}\psi^{(\leq k)+}_{\mathbf{k},\omega,\varepsilon\sigma}\psi^{(\leq k)-}_{\mathbf{k}+\mathbf{p},\omega,\varepsilon\sigma}.\end{aligned} \tag{113}$$

Proceeding as in Sect. 2 we can write

$$\hat{W}^{(k)}_{\Delta}(\mathbf{p};\underline{\mathbf{k}}) = \sum_{n=0}^{\infty}\sum_{\tau\in\mathcal{T}^{2,1}_{k,n}}\sum_{\substack{\mathbf{P}\in\mathcal{P}\\ |P_{v_0}|=2}}\hat{W}^{(k)}_{\Delta,\tau}(\mathbf{p};\underline{\mathbf{k}}), \tag{114}$$

where $\mathcal{T}^{2,1}_{j,n}$ is a family of trees, defined as in Sect. 2 with the only difference that to the end-points v is now associated (111) for $h_v\geq 0$ or (113) for $h_v<0$.

Assume that

$$|\nu^{\varepsilon}_k| \leq C|\lambda|\gamma^{-\frac{1}{2}(N-k)}, \tag{115}$$

then

$$\| W^{(k)}_{\Delta,\tau} \|_k \le (c\bar{\varepsilon}_h)^n \gamma^{-\frac{1}{2}(N-k)} \gamma^{h(2-\frac{|P_{v_0}|}{2}-n^\alpha_{v_0})} \times \prod_{v \text{ not e.p.}} \gamma^{-(\frac{|P_v|}{2}-2+z_v+n^\alpha_v)} \tag{116}$$

where n^α_v is the number of endpoints of type α following the vertex v and, by construction, $n^\alpha_{v_0} = 1$. $\frac{|P_v|}{2} - 2 + z_v + n^\alpha_v > 0$; this formula implies immediately (109).

The bound (104) says that, for obtaining the dimensional bound (116), the function $S^{(i,j)}$ is exactly equivalent to the contraction of the operator $J\psi^+\psi^-$, with one ψ field contracted on scale i, and the other contracted on scale j. This is coherent with thinking to the external field $D_\omega(\mathbf{p})\hat{\alpha}_{\mathbf{p},\omega,\sigma}$ as bearing the same dimension of the J field.

To avoid the $(n!)^2$ bounds for the truncated expectation require more care. Indeed, in the contraction of the operator $J\psi^+\psi^-$ one propagator belongs to the anchored tree of formula (17), while the other may belong to the anchored tree, or be inside the Gram determinant. When studying the contraction of the kernel T_0 it is convenient to avoid the bound of the Gram determinant with (24) directly. The determinant can be expanded with respect to the entries of one the row and the corresponding minors; in particular, we choose the row (made of l entries) containing the propagator coming out of the operator T_0, so that, together with the other propagator in the anchored tree, we can reconstruct the function $S^{(i,j)}$ times a monomial in the parameters $\mathbf{t}$ that can be always bounded by 1; the corresponding minors are Gram determinants of dimension $l-1$, that can be bounded as in (11). Therefore, the expansion with respect to a row make us loose a factor l with respect to the usual bound, namely a C^n factor more in the final bound.

In order to prove (115) we note that

$$\begin{aligned} &W^{(1;2)(k)}_{\Delta;\bar{\omega},\omega,\sigma,\sigma'}(\mathbf{z};\mathbf{x},\mathbf{y}) \\ &\quad = \sum_{i,j=k}^{N} \int d\mathbf{u} d\mathbf{w} S^{(i,j)}_{\bar{\omega},\omega}(\mathbf{z};\mathbf{u},\mathbf{w}) W^{(0;4)(k)}_{\omega;\sigma,\sigma'}(\mathbf{u},\mathbf{w},\mathbf{x},\mathbf{y}) \\ &\qquad - \delta_{\bar{\omega},-\omega}\nu_N \int d\mathbf{w} v(\mathbf{z}-\mathbf{w}) W^{(1;2)(k)}_{\omega;-\sigma,\sigma'}(\mathbf{w};\mathbf{x},\mathbf{y}). \end{aligned} \tag{117}$$

The reason for which in the second line of (117) there is $W^{(0;4)(k)}_{\omega;\sigma,\sigma'}(\mathbf{u},\mathbf{w},\mathbf{x},\mathbf{y})$ and not also non-connected graphs whit four external legs is the following:

(a) Defining $(1-\chi_N(\mathbf{k}))f_i(\mathbf{k}) = \delta_{i,N}u_N(\mathbf{k})$, the graphs in which one between the fields $\hat{\psi}$ in T_0 is contracted with a kernel $\hat{W}^{(0;2)(k)}_\sigma$ is of the form:

$$\frac{\chi_N(\mathbf{k}+\mathbf{p})-1}{\chi_N(\mathbf{k}+\mathbf{p})} D_\omega(\mathbf{k}+\mathbf{p})\hat{g}_\omega(\mathbf{k})\hat{W}^{(0;2)(k)}_\sigma(\mathbf{k}) - u_N(\mathbf{k})\hat{W}^{(0;2)(k)}_\sigma(\mathbf{k}). \tag{118}$$

This term is not compatible with the structure of the multiscale expansion of the Schwinger functions, since by support properties we have $|\mathbf{k}+\mathbf{p}|, |\mathbf{k}| > \gamma^N$ while, by construction, the fields $\psi^{\le N-1}_{\mathbf{k}+\mathbf{p}}$ and $\psi^{\le N-1}_{\mathbf{k}}$, implies the constraint $|\mathbf{k}+\mathbf{p}|, |\mathbf{k}| < \gamma^N$.

(b) The graphs in which all and two the fields $\hat{\psi}$ in T_0 are contracted, each one with its own $\hat{W}^{(0;2)(k)}_\sigma$ have the form

$$-[u_N(\mathbf{k}+\mathbf{p})\hat{g}_\omega(\mathbf{k}) - u_N(\mathbf{k})\hat{g}_\omega(\mathbf{k}+\mathbf{p})]$$

$$\times \hat{W}^{(0;2)(k)}_{\omega}(\mathbf{k})\hat{W}^{(0;2)(k)}_{\sigma}(\mathbf{k}+\mathbf{p}) \tag{119}$$

hence they are not compatible with the multiscale expansion for the very same reason as above.

Equation (117) is analyzed in a way similar to the one followed in Sect. 3; by using the decomposition of $W^{(0;4)(k)}_{\omega,\sigma}$ in Fig. 8, so obtaining the decomposition for $W^{(1;2)(k)}_{\Delta;\bar{\omega},\omega,\sigma,\sigma'}$ depicted in Fig. 9.

Fixed the integer q and calling $b_j(\mathbf{x}) \overset{def}{=} C_q\gamma^j/(1+[\gamma^j|\mathbf{x}|]^q)$, we bound the r.h.s. member in the same spirit as in Sect. 3.

1. Graphs (c) and (d) are:

$$\sum_{i,j=k}^{N}\int d\mathbf{u}d\mathbf{u}'d\mathbf{w}d\mathbf{w}'S^{(i,j)}_{-\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{w})g_\omega(\mathbf{u}-\mathbf{u}')v(\mathbf{u}-\mathbf{w}')$$
$$\times W^{(1;4)(k)}_{\omega;-\sigma,\sigma,\sigma'}(\mathbf{w}';\mathbf{u}',\mathbf{w},\mathbf{x},\mathbf{y})$$
$$+\sum_{i,j=k}^{N}\int d\mathbf{u}d\mathbf{u}'d\mathbf{w}d\mathbf{w}'S^{(i,j)}_{-\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{w})W^{(0;2)(k)}_{\omega;\sigma}(\mathbf{w},\mathbf{w}')$$
$$\times g_\omega(\mathbf{w}'-\mathbf{u})v(\mathbf{u}-\mathbf{u}')W^{(1;2)(k)}_{\omega;-\sigma,\sigma'}(\mathbf{u}';\mathbf{x},\mathbf{y}). \tag{120}$$

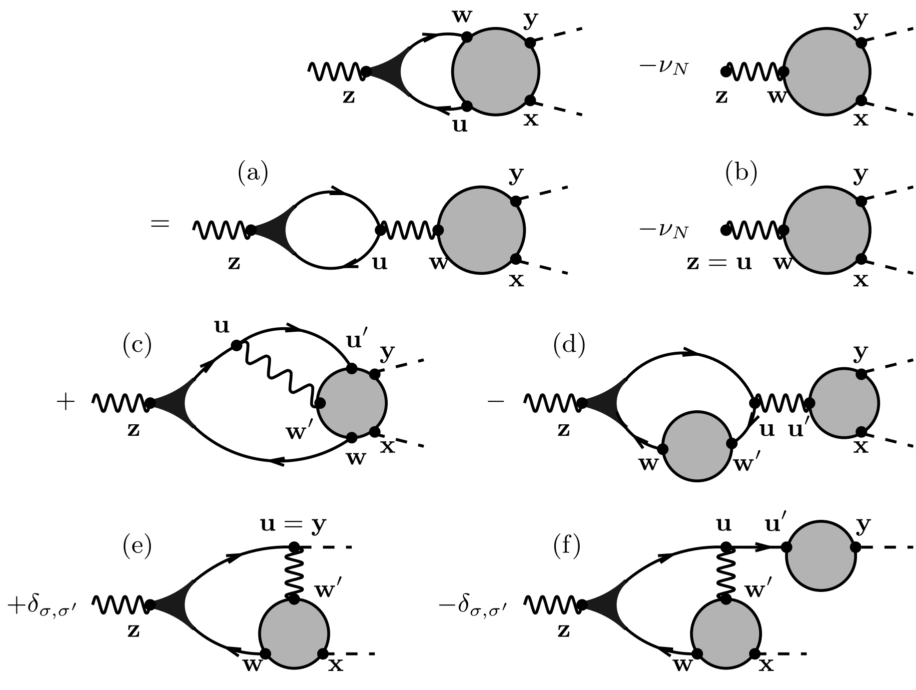


Fig. 9 Graphical representation of (112)

Since either i or j has to be N, and by the bound (104), the norm of (c) is bounded by

$$2|v|_\infty \sum_{j,m=k}^{N} \int d\mathbf{x} d\mathbf{u}' d\mathbf{w} d\mathbf{w}' |W^{(1;4)(k)}_{\omega;-\sigma,\sigma,\sigma'}(\mathbf{w}'; \mathbf{u}', \mathbf{w}, \mathbf{x}, \mathbf{y})| \times \int d\mathbf{z} d\mathbf{u} b_N(\mathbf{z}-\mathbf{u}) b_j(\mathbf{z}-\mathbf{w}) |g^{(m)}_\omega(\mathbf{u}-\mathbf{u}')| \tag{121}$$

and hence we can clearly proceed as for (66) but now the scale of higher momenta is fixed to be N, and therefore we get the bound

$$C_1|\lambda| \cdot |v|_\infty \cdot \gamma^{-k} \sum_{i=k}^{N} \sum_{i'=k}^{i} \gamma^{-N} \gamma^{-i} \gamma^{i'} \le C_2|\lambda|\gamma^{-k-N}(N-k) \le C_3|\lambda|\gamma^{-2k}\gamma^{-(1/2)(N-k)}. \tag{122}$$

A similar bound can be obtained for (d).

2. The graphs (e) and (f) are:

$$\delta_{\sigma,\sigma'} \sum_{i,j=k}^{N} \int d\mathbf{u} d\mathbf{w} d\mathbf{w}' S^{(i,j)}_{-\omega,\omega}(\mathbf{z}; \mathbf{u}, \mathbf{w}) W^{(1;2)(k)}_{\omega;\sigma,\sigma}(\mathbf{w}'; \mathbf{w}, \mathbf{x}) \times \left[\delta(\mathbf{u}-\mathbf{y}) + \int d\mathbf{u}' g_\omega(\mathbf{u}-\mathbf{u}') v(\mathbf{y}-\mathbf{w}') W^{(0;2)(k)}_{\omega;\sigma,\sigma}(\mathbf{u}', \mathbf{y})\right]. \tag{123}$$

The bound for the graph (e) is $C|v|_\infty \cdot \|W^{(1;2)(k)}_{\omega;\sigma,\sigma}\|_k \cdot |b_N|_1 \sum_{j=k}^N |b_j|_1 \le C\gamma^{-(N-k)}\gamma^{-2k}$. Similar bound holds for (f).

3. The graphs (a) and (b) are:

$$\int d\mathbf{u} \left[\sum_{i,j=k}^{N} S^{(i,j)}_{-\omega,\omega}(\mathbf{z}; \mathbf{u}, \mathbf{u}) - \nu_N \delta(\mathbf{z}-\mathbf{u})\right] \times \int d\mathbf{w} v(\mathbf{u}-\mathbf{w}) W^{(1;2)(k)}_{\omega;-\sigma,\sigma'}(\mathbf{w}, \mathbf{x}, \mathbf{y}). \tag{124}$$

Using the identity (69), for graph (a) we have

$$\begin{aligned}
&\sum_{i,j=k}^{N} \int d\mathbf{u} d\mathbf{w} S^{(i,j)}_{-\omega,\omega}(\mathbf{z}; \mathbf{u}, \mathbf{u}) v(\mathbf{u}-\mathbf{w}) W^{(1;2),(k)}_{\omega;-\sigma,\sigma'}(\mathbf{w}; \mathbf{x}, \mathbf{y}) \\
&= \int d\mathbf{w} v(\mathbf{z}-\mathbf{w}) W^{(1;2),(k)}_{\omega;-\sigma,\sigma'}(\mathbf{w}; \mathbf{x}, \mathbf{y}) \sum_{i,j=k}^{N} \int d\mathbf{u} S^{(i,j)}_{-\omega,\omega}(\mathbf{z}; \mathbf{u}, \mathbf{u}) \\
&\quad + \sum_{p=0,1} \sum_{i,j=k}^{N} \int d\mathbf{u} S^{(i,j)}_{-\omega,\omega}(\mathbf{z}; \mathbf{u}, \mathbf{u})(u_p - z_p) \\
&\quad \times \int_0^1 d\tau \int d\mathbf{w} (\partial_p v)(\mathbf{z}-\mathbf{w}+\tau(\mathbf{u}-\mathbf{z})) W^{(1;2),(k)}_{\omega';-\sigma,\sigma'}(\mathbf{w}; \mathbf{x}, \mathbf{y}).
\end{aligned} \tag{125}$$

The latter term is irrelevant and vanishing in the limit $N-k\to+\infty$: using that one between i and j is on scale N, a bound for its norm is

$$2\|W^{(1;2),(k)}_{\omega';-\sigma,\sigma'}\|_k\cdot|\partial v|_1\cdot\sum_{j=k}^{N}\int d\mathbf{u}b_N(\mathbf{z}-\mathbf{u})b_j(\mathbf{z}-\mathbf{u})|(u_p-z_p)| \tag{126}$$

and we obtain the bound $C\gamma^{-k}\gamma^{-(N-k)}$. The former term in the r.h.s. member of (125) is compensated by (b). Indeed we have

$$\sum_{i,j=k}^{N}\int d\mathbf{u}S^{(i,j)}_{-\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{u})-\nu_N=2\sum_{j\le k-1}\int d\mathbf{u}S^{(N,j)}_{-\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{u}) \tag{127}$$

and hence the bound for such a difference is $C\gamma^{-(N-k)}$.

The graph expansion for $W^{(1;2)(k)}_{\Delta;\omega,\omega,\sigma,\sigma'}$ is again given by Fig. 9, but for ν_N replaced by 0. Hence a bound can be obtained with the same above argument, with only one important difference: the contribution that in the previous analysis were compensated by (b) now are zero by symmetries. Indeed, calling $\mathbf{k}^*$ the rotation of $\mathbf{k}$ of $\pi/2$ and since $\hat S^{(i,j)}_{\bar\omega,\omega}(\mathbf{k}^*,\mathbf{p}^*)=-\omega\bar\omega\hat S^{(i,j)}_{\bar\omega,\omega}(\mathbf{k},\mathbf{p})$, in place of the bound (127), in this case we have:

$$\sum_{i,j=k}^{N}\int d\mathbf{u}S^{(i,j)}_{\omega,\omega}(\mathbf{z};\mathbf{u},\mathbf{u})=\sum_{i,j=k}^{N}\int\frac{d\mathbf{k}}{(2\pi)^2}\hat S^{(i,j)}_{\omega,\omega}(\mathbf{k},-\mathbf{k})=0. \tag{128}$$

Finally, so far we have obtained (115) for $k\ge0$.

Let us consider, now, the case $k<0$. By (112) we have

$$\begin{aligned}
&\mathcal{L}\hat W^{(1;2)(k)}_{\Delta;\bar\omega,\omega,\sigma,\sigma'}(\mathbf{p};\mathbf{k})\\
&=\hat W^{(1;2)(k)}_{\Delta;\bar\omega,\omega,\sigma,\sigma'}(0;0)\\
&=\int\frac{d\mathbf{q}}{(2\pi)^2}\hat S^{(i,j)}_{\bar\omega,\omega}(\mathbf{q},\mathbf{q})\hat W^{(0;4)(k)}_{\omega;\sigma,\sigma'}(\mathbf{q},\mathbf{q},0)-\delta_{\bar\omega,-\omega}\nu_N\hat W^{(1;2)(k)}_{\omega;-\sigma,\sigma'}(0,0).
\end{aligned} \tag{129}$$

As we noticed in Sect. 2 $\hat W^{(1;2)(k)}_{\omega;-\sigma,\sigma'}(0,0)=\delta_{-\sigma,\sigma'}$; furthermore, under a rotation of $\pi/2p$,

$$\begin{aligned}
&\hat S^{(i,j)}_{\bar\omega,\omega}(\mathbf{p}^*,\mathbf{q}^*)=e^{-i(\omega+\bar\omega)\frac{\pi}{2p}}\hat S^{(i,j)}_{\bar\omega,\omega}(\mathbf{p},\mathbf{q}),\\
&\hat W^{(0;4)(k)}_{p,\underline{\omega}}(\mathbf{p}^*,\mathbf{q}^*,0)=e^{-i\omega\pi(1-\frac{1}{p})}\hat W^{(0;4)(k)}_{p,\underline{\omega}}(\mathbf{p},\mathbf{q},0)
\end{aligned} \tag{130}$$

hence the integral in (129) is non-zero only for $p=1$ and $\bar\omega=-\omega$, case in which (129) is reduces to (127). □

We can finally discuss the bound for $R^{(1;2)}_{\omega;\sigma',\sigma}(\mathbf{p};\mathbf{k})$ so finally proving Lemma 6. It can be written by a sum of trees essentially identical to the ones for $\hat G^{(1;2)}_{\omega;\sigma',\sigma}(\mathbf{p};\mathbf{k})$, with the only important difference that there are three different special endpoints associated to the field α, corresponding to the three different terms in (100); we call these endpoints of type T_+,T_-,T_0 respectively.

The sum over the trees such that the endpoint is of type $\nu^{\pm}_{k,\omega,\sigma}$ can be bounded as in (94), the only difference being that, thanks to the bound (115), one has to multiply the r.h.s. by a factor $|\lambda|\gamma^{-\frac{1}{2}(N-k)}$, for k the scale of the endpoint. This factor has to be inserted also in the r.h.s. of the bounds (95), hence, it is easy to see that the contributions of these trees vanishes as $N \to \infty$.

Let us now consider the trees with an endpoint of type T_0. The fields of the T_0 endpoint are contracted at scale j, N; this implies that $h_J = N$: since $d_v + r_v - 1/4 > 0$ for all vertices belonging to the path connecting the endpoint to the root, we can replace in the r.h.s. of the bounds (95) $d_v + r_v$ with $d_v + r_v - 1/4$ and add a factor $\gamma^{-(N-h_{\mathbf{k}})/4}$, so that

$$\lim_{N\to\infty} R^{(1;2)}_{\omega;\sigma',\sigma}(\mathbf{p};\mathbf{k}) = 0. \tag{131}$$

6 The Closed Equation

By (54), for $k = -\infty$, we obtain the Schwinger-Dyson equation for the two-point Schwinger function

$$\hat{S}_{N;\omega,\sigma}(\mathbf{k}) = \hat{g}_\omega(\mathbf{k})\left[1 - \lambda \int \frac{d\mathbf{p}}{(2\pi)^2}\hat{v}(\mathbf{p})\hat{G}_{N;\omega,-\sigma;\sigma}(\mathbf{p};\mathbf{k})\right]. \tag{132}$$

We define

$$a_N(\mathbf{p}) = \frac{1}{D_\omega(\mathbf{p}) - \nu_N \hat{v}(\mathbf{p}) D_{-\omega}(\mathbf{p})}, \qquad \bar{a}_N(\mathbf{p}) = \frac{1}{D_\omega(\mathbf{p}) + \nu_N \hat{v}(\mathbf{p}) D_{-\omega}(\mathbf{p})}$$

and summing over σ' the equation, we obtain the *vector Ward Identity* (associated the phase symmetry):

$$\sum_{\sigma'} \hat{G}_{N;\omega,\sigma';\sigma}(\mathbf{p};\mathbf{k}) = a_N(\mathbf{p}) \sum_{\varepsilon,\bar{\omega}} D_{\bar{\omega}}(\mathbf{p}) \hat{R}^{(1;2)}_{\bar{\omega},\omega;\varepsilon\sigma,\sigma}(\mathbf{p};\mathbf{k}) + a_N(\mathbf{p})[\hat{S}_{N;\omega,\sigma}(\mathbf{k}) - \hat{S}_{N;\omega,\sigma}(\mathbf{k}+\mathbf{p})] \tag{133}$$

while multiplied times σ' the equation, and summing over σ', we obtain the *axial Ward Identity* (associated to the chiral symmetry):

$$\sum_{\sigma'} \sigma' \hat{G}_{N;\omega,\sigma';\sigma}(\mathbf{p};\mathbf{k}) = \sigma \bar{a}_N(\mathbf{p}) \sum_{\varepsilon,\bar{\omega}} \varepsilon D_{\bar{\omega}}(\mathbf{p}) \hat{R}^{(1;2)}_{\bar{\omega},\omega;\varepsilon\sigma,\sigma}(\mathbf{p};\mathbf{k}) + \sigma \bar{a}_N(\mathbf{p})[\hat{S}_{N;\omega,\sigma}(\mathbf{k}) - \hat{S}_{N;\omega,\sigma}(\mathbf{k}+\mathbf{p})]. \tag{134}$$

Finally, from these two equations, since $\frac{1+\rho\sigma'}{2} = \delta_{\rho,\sigma'}$

$$\hat{G}_{N;\omega,\sigma';\sigma}(\mathbf{p};\mathbf{k}) = \sum_{\varepsilon,\bar{\omega}} \frac{a_N(\mathbf{p}) + \bar{a}_N(\mathbf{p})\varepsilon}{2} D_{\bar{\omega}}(\mathbf{p}) \hat{R}^{(1;2)}_{\bar{\omega},\omega;\varepsilon\sigma',\sigma}(\mathbf{p};\mathbf{k}) + \frac{a_N(\mathbf{p}) + \sigma\sigma' \bar{a}_N(\mathbf{p})}{2}[\hat{S}_{N;\omega,\sigma}(\mathbf{k}) - \hat{S}_{N;\omega,\sigma}(\mathbf{k}+\mathbf{p})]. \tag{135}$$

In order to shorten the notation we now define

$$\hat{A}_\varepsilon(\mathbf{p}) \stackrel{def}{=} \frac{\hat{v}(\mathbf{p})[a(\mathbf{p}) + \varepsilon \bar{a}(\mathbf{p})]}{2}. \tag{136}$$

Let

$$\hat{R}^{(2)}_{\omega;\varepsilon;\sigma}(\mathbf{k}) \stackrel{def}{=} \sum_{\bar{\omega}} \int \frac{d\mathbf{p}}{(2\pi)^2} \bar{\chi}_N(\mathbf{p}) \hat{A}_-(\mathbf{p}) D_{\bar{\omega}}(\mathbf{p}) \hat{R}^{(1;2)}_{\bar{\omega},\omega;-\varepsilon\sigma;\sigma}(\mathbf{p};\mathbf{k})$$

Theorem 4 *If* $|\lambda|$ *is small enough and for fixed momentum* $\mathbf{k}$, *in the limit* $N \to \infty$ *we obtain*

$$D_\omega(\mathbf{k}) \hat{S}_{\omega,\sigma}(\mathbf{k}) = 1 + \lambda \int \frac{d\mathbf{p}}{(2\pi)^2} \hat{A}_-(\mathbf{p}) \hat{S}_{\omega,\sigma}(\mathbf{k}+\mathbf{p}). \tag{137}$$

By solving (137) (see Appendix B) and using (88), Theorem 1 follows. In order to prove of Theorem 4 we have to show that

$$\lim_{N\to\infty} \hat{R}^{(2)}_{\omega;\varepsilon;\sigma}(\mathbf{k}) = 0 \tag{138}$$

it is convenient to write

$$\hat{R}^{(0;2)}_{\omega;\varepsilon;\sigma}(\mathbf{k}) = \frac{\partial^2 \mathcal{W}_{T,\varepsilon}}{\partial \hat{\beta}_{\mathbf{k},\omega,\sigma} \partial \hat{\varphi}^-_{\mathbf{k},\omega,\sigma}}(0) \tag{139}$$

where we have introduced the new generating functional

$$\begin{aligned} e^{\mathcal{W}_{T,\varepsilon}(\beta,\varphi)} &= \int P(d\psi^{(\le N)}) e^{-\mathcal{V}^{(N)}_{T,\varepsilon}(\psi^{\le N},\beta,\varphi)} \\ &\stackrel{def}{=} \int P(d\psi) \exp\{-\lambda V(\psi^{(\le N)}) + [T_1^{(\varepsilon)} - \nu_N T_-^{(\varepsilon)}](\psi^{(\le N)},\beta)\} \\ &\quad \times \exp\left\{\sum_{\omega,\sigma} \int d\mathbf{x} [\varphi^+_{\mathbf{x},\omega,\sigma} \psi^{(\le N)-}_{\mathbf{x},\omega,\sigma} + \psi^{(\le N)+}_{\mathbf{x},\omega,\sigma} \varphi^-_{\mathbf{x},\omega,\sigma}]\right\} \end{aligned} \tag{140}$$

with

$$\begin{aligned} T_1^{(\varepsilon)}(\psi,\beta) &= \sum_{\omega,\sigma} \int \frac{d\mathbf{p}d\mathbf{q}}{(2\pi)^4} \bar{\chi}_N(\mathbf{p}) \hat{A}_\varepsilon(\mathbf{p}) C_{N;\omega}(\mathbf{q}+\mathbf{p},\mathbf{q}) \\ &\quad \times \hat{\beta}_{\mathbf{k},\omega,\sigma} \hat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega,\sigma} \hat{\psi}^+_{\mathbf{q}+\mathbf{p},\omega,-\varepsilon\sigma} \hat{\psi}^-_{\mathbf{q},\omega,-\varepsilon\sigma}, \\ T_-^{(\varepsilon)}(\psi,\beta) &= \sum_{\omega,\sigma} \int \frac{d\mathbf{p}d\mathbf{q}}{(2\pi)^4} \bar{\chi}_N(\mathbf{p}) \hat{A}_\varepsilon(\mathbf{p}) \hat{v}(\mathbf{p}) D_{-\omega}(\mathbf{p}) \\ &\quad \times \hat{\beta}_{\mathbf{k},\omega,\sigma} \hat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega,\sigma} \hat{\psi}^+_{\mathbf{q}+\mathbf{p},\omega,\varepsilon\sigma} \hat{\psi}^-_{\mathbf{q},\omega,\varepsilon\sigma} \end{aligned} \tag{141}$$

and ν_N is defined in the previous section.

The integration of $\mathcal{W}_{T,\varepsilon}$ can be done in a way very similar to the previous ones. After the integration of the fields $\psi^{(N)}, \ldots, \psi^{(k+1)}$, we get

$$e^{-\mathcal{V}^{(k)}_{T,\varepsilon}(\psi^{(\le k)},\beta,\varphi)} \stackrel{def}{=} \int P(d\psi^{[k+1,N]}) e^{-\mathcal{V}^{(N)}_{T,\varepsilon}(\psi^{(\le N)},\beta,\varphi)} \tag{142}$$

and we call $\mathcal{H}^{(k)}_{T,\varepsilon}$ the part of $\mathcal{V}^{(k)}_{T,\varepsilon}$ that is linear in β

$$\begin{aligned} &\mathcal{H}^{(k)}_{T,\varepsilon}(\psi,0,\beta) \\ &\quad = \sum_{m\ge 1} \sum_{\omega,\underline{\sigma}} \int d\mathbf{z} d\mathbf{x} d\mathbf{y} \frac{H^{(1;2m+1)(k)}_{T,\varepsilon;\omega,\underline{\sigma}}(\mathbf{z};\mathbf{x},\mathbf{y})}{2m!} \beta_{\mathbf{z},\omega,\sigma'} \prod_{i=1}^{m} \psi^+_{\mathbf{x}_i,\omega,\sigma_i} \prod_{i=1}^{m+1} \psi^-_{\mathbf{y}_i,\omega,\sigma_i}. \end{aligned} \tag{143}$$

Theorem 5 *If* $|\lambda|$ *small enough, for any* $h: k+1 \le h \le N$,

$$\|H^{(1;2m+1)(k)}_{T,\varepsilon;\underline{\sigma},\underline{\omega}}\|_k \le C\gamma^{-\frac{1}{2}(N-k)}\gamma^{k(1-m)} \tag{144}$$

Proof The integration is done exactly as in Sect. 2; we define for $0 \le k \le N$

$$\begin{aligned} \mathcal{L}\hat{H}^{(1;1)(k)}_{T,\varepsilon;\underline{\sigma},\underline{\omega}}(\mathbf{p};\mathbf{k}) &= \hat{H}^{(1;1)(k)}_{T,\varepsilon;\underline{\sigma},\underline{\omega}}(\mathbf{p};\mathbf{k}) \stackrel{def}{=} \hat{z}^{\varepsilon}_k(\mathbf{p};\mathbf{k}),\\ \mathcal{L}\hat{H}^{(1;3)(k)}_{T,\varepsilon;\underline{\sigma},\underline{\omega}}(\mathbf{p};\underline{\mathbf{k}}) &= \hat{H}^{(1;3)(k)}_{T,\varepsilon;\underline{\sigma},\underline{\omega}}(\mathbf{p};\underline{\mathbf{k}}) \stackrel{def}{=} \hat{\lambda}^{\varepsilon}_k(\mathbf{p};\underline{\mathbf{k}}) \end{aligned} \tag{145}$$

so that for $k \ge 0$

$$\begin{aligned} \mathcal{L}\mathcal{V}^{(k)}_{\mathcal{T},\varepsilon} &= \mathcal{L}\mathcal{V}(\psi^{(\le k)},0)\\ &\quad + \int \frac{d\mathbf{k}d\mathbf{p}d\mathbf{q}}{(2\pi)^4}\hat{\lambda}^{\varepsilon}_k(\mathbf{k},\mathbf{p},\mathbf{q})\hat{\beta}_{\mathbf{p},\omega,\sigma}\hat{\psi}^{(\le k)-}_{\mathbf{k}+\mathbf{p},\omega,\sigma}\hat{\psi}^{(\le k)+}_{\mathbf{k},\omega,-\sigma}\hat{\psi}^{(\le k)-}_{\mathbf{k}+\mathbf{p},\omega,-\sigma}\\ &\quad + \int \frac{d\mathbf{k}}{(2\pi)^2}\hat{z}^{\varepsilon}_k(\mathbf{k})\hat{\beta}_{\mathbf{k},\omega,\sigma}\hat{\psi}^{(\le k)-}_{\mathbf{k},\omega,\sigma} \end{aligned} \tag{146}$$

where $\mathcal{L}\mathcal{V}(\psi^{(\le k)},0)$ is given by the first two addenda of (38).

On the other hand for $k \le 0$ we define

$$\begin{aligned} \mathcal{L}\hat{H}^{(1;1)(k)}_{T,\varepsilon;\underline{\sigma},\underline{\omega}}(\mathbf{p};\mathbf{k}) &= \hat{H}^{(1;1)(k)}_{T,\varepsilon;\underline{\sigma},\underline{\omega}}(0;0) \equiv \tilde{z}^{\varepsilon}_k,\\ \mathcal{L}\hat{H}^{(1;3)(k)}_{T,\varepsilon;\underline{\sigma},\underline{\omega}}(\mathbf{p};\underline{\mathbf{k}}) &= \hat{H}^{(1;3)(k)}_{T,\varepsilon;\underline{\sigma},\underline{\omega}}(0;0) = \tilde{\lambda}^{\varepsilon}_k \end{aligned} \tag{147}$$

so that for $h < 0$

$$\begin{aligned} \mathcal{L}\mathcal{V}^{(k)}_{\mathcal{T},\varepsilon} &= \mathcal{L}\mathcal{V}(\psi^{(\le k)},0)\\ &\quad + \hat{\lambda}^{\varepsilon}_k \int \frac{d\mathbf{k}d\mathbf{p}d\mathbf{q}}{(2\pi)^4}\hat{\beta}_{\mathbf{p},\omega,\sigma}\hat{\psi}^{(\le k)-}_{\mathbf{k}+\mathbf{p},\omega,\sigma}\hat{\psi}^{(\le k)+}_{\mathbf{k},\omega,-\sigma}\hat{\psi}^{(\le k)-}_{\mathbf{k}+\mathbf{p},\omega,-\sigma}\\ &\quad + \hat{z}^{\varepsilon}_k \int \frac{d\mathbf{k}}{(2\pi)^2}\hat{\beta}_{\mathbf{k},\omega,\sigma}\hat{\psi}^{(\le k)-}_{\mathbf{k},\omega,\sigma} \end{aligned} \tag{148}$$

where $\mathcal{L}\mathcal{V}(\psi^{(\le k)},0)$ is given by the first two addenda of (39). Proceeding as in Sect. 2 we can write

$$H^{(1;2m+1)(k)}_{T,\varepsilon;\underline{\sigma},\underline{\omega}} = \sum_{n=0}^{\infty}\sum_{\tau\in\mathcal{T}_{k,n}}\sum_{\mathbf{P}} H^{(1;2m+1)(k)}_{T,\varepsilon;\tau,\mathbf{P}} \tag{149}$$

where $\mathcal{T}_{k,n}$ is a family of trees, defined as in Sect. 2 with the only difference that to the end-points v is now associated (146) for $h_v \ge 0$ or (148) for $h_v < 0$; and there is one special endpoint with field β.

Assume that, for any k,

$$\|\lambda^{\varepsilon}_k\|_k, \|z^{\varepsilon}_k\|_k \le C|\lambda|\gamma^{-\frac{1}{2}(N-k)}, \tag{150}$$

then, proceeding as above

$$\|H^{(k)}_{T,\varepsilon;\tau,\mathbf{P}}\|_k \le (c\bar{\varepsilon}_{k+1})^{n-n^{\alpha}_{v_0}}\gamma^{-\frac{1}{2}(N-k)}\gamma^{h(2-\frac{|P_{v_0}|}{2}-n^{\alpha}_{v_0})}$$

$$\times \prod_{v \text{ not e.p.}} \gamma^{-(\frac{|P_v|}{2}-2+z_v+n_v^\alpha)} \tag{151}$$

and again $\frac{|P_v|}{2} - 2 + z_v + n_v^\alpha > 0$. In order to prove (150) we can write

$$H^{(1;3)(k)}_{T,\varepsilon;\omega,\omega',\sigma} = H^{a(1;3)(k)}_{T,\varepsilon;\omega,\omega',\sigma} + H^{b(1;3)(k)}_{T,\varepsilon;\omega,\omega',\sigma} \tag{152}$$

where:

1. $H^{a(1;3)(k)}_{T,\varepsilon;\omega,\underline{\sigma}}$ contains the term in which the field $\hat\psi_{\mathbf{k+p},\omega,\sigma}$ of T_1 and T_- is not contracted or is contracted with a $\hat W^{(0;2)(k)}$:

$$\begin{aligned}&\hat H^{a(1;3)(k)}_{T,\varepsilon;\omega,\underline{\sigma}}(\mathbf{k},\mathbf{p})\\ &\quad= [1+\hat g^{[k+1,N]}_\omega(\mathbf{p})\hat W^{(0;2)(k)}(\mathbf{p})]\hat A_\varepsilon(\mathbf{p})D_\omega(\mathbf{p})\hat W^{(1;2)(k)}_{\Delta,\varepsilon;\omega,\omega',\sigma}(\mathbf{k};\mathbf{p})\end{aligned} \tag{153}$$

   for $k \ge 0$ we have already proved the bound $\|W^{(0;2)(k)}\|_k \le C|\lambda|^2\gamma^{-k}$; for $k<0$, we use the fact that the local part of (up to first order of Taylor expansion in $\mathbf{k}$) $\hat W^{(0;2)(k)}$ is zero, and the rest has a dimensional gain of one degree; by (109),

$$\|H^{a(1;3)(k)}_{T,\varepsilon;\omega,\underline{\sigma}}\|_k \le C|\lambda|\gamma^{-\frac12(N-k)}. \tag{154}$$

   For $k \le 0$ we have defined $\mathcal{L}\hat H^{a(1;3)(k)}_{T,\varepsilon;\omega,\underline{\sigma}}(\mathbf{k},\mathbf{p}) = \hat H^{a(1;3)(k)}_{T,\varepsilon;\omega,\underline{\sigma}}(0,0)$ and we know by symmetry that $\hat H^{a(1;3)(k)}_{T,\varepsilon;\omega,\underline{\sigma}}(0,0)=0$.
2. $H^{b(1;3)(k)}_{T,\varepsilon;\omega,\underline{\sigma}}$ contains the term in which the field $\hat\psi_{\mathbf{k+p},\omega,\sigma}$ of T_1 and T_- is contracted. We can further distinguish them as in Fig. 10; we can write

$$\begin{aligned}&H^{b(1;3)(k)}_{T,\varepsilon;\omega,\underline{\sigma}}(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v})\\ &\quad= \int d\mathbf{z}d\mathbf{w}\bar v(\mathbf{x}-\mathbf{z})g^{[k+1,N]}_\omega(\mathbf{x}-\mathbf{w})K^{(1;4)(k)}_{\Delta,\varepsilon;\omega,\omega',\sigma}(\mathbf{z};\mathbf{w},\mathbf{y},\mathbf{u},\mathbf{v})\end{aligned} \tag{155}$$

   where

$$\bar v(\mathbf{x}) = \int d\mathbf{p}\, e^{i\mathbf{px}}\hat A_\varepsilon(\mathbf{p})D_\omega(\mathbf{p})$$

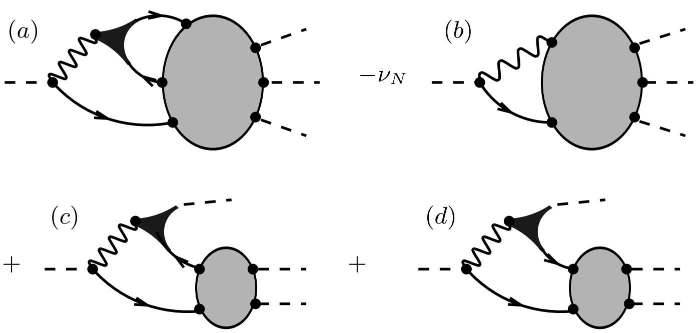


Fig. 10 Graphical representation of $H^{b(1;3)(k)}_{T,\varepsilon;\omega,\underline{\sigma}}$

so that, by the bounds for $\|K^{(1;4)(k)}_{\Delta,\varepsilon;\omega,\omega',\sigma}\|_k$, $|\bar v|_\infty$ and $|g^{(j)}_\omega|_1$,

$$\|H^{b(1;3)(k)}_{T,\varepsilon;\omega,\underline{\sigma}}\|_k \le C|\lambda|\gamma^{-k}\gamma^{\frac12(N-k)}. \quad (156)$$

While for $k<0$ we have that the local part of the graph is zero by transformation under rotation.

We consider now the terms contributing to $H^{(1;1)(k)}_{T,\varepsilon;\omega,\sigma}$.

1. The contraction of the field $\hat\psi^+_{\mathbf{q}+\mathbf{p},\omega,-\varepsilon\sigma}$ with $\hat\psi^-_{\mathbf{q},\omega,-\varepsilon\sigma}$ of T_1, possibly through a kernel $\hat W^{(0;2)(k)}(\mathbf{q})$, can only happen for $\mathbf{p}=0$, and therefore it is forbidden by $\bar\chi_N(\mathbf{p})$.
2. The contraction of $\hat\psi^+_{\mathbf{q}+\mathbf{p},\omega,-\varepsilon\sigma}$ with $\hat\psi^-_{\mathbf{k}+\mathbf{p},\omega,\sigma}$ (that can take place only for $\varepsilon=-$), possibly through a kernel $\hat W^{(0;2)(k)}(\mathbf{q}+\mathbf{p})$, and possibly with $\hat\psi^-_{\mathbf{q},\omega,-\varepsilon\sigma}$ contracted with a second kernel $\hat W^{(0;2)(k)}(\mathbf{q})$, has the following expression

$$\hat H^{a(1;1)(k)}_{T,\varepsilon;\omega,\sigma}(\mathbf{k}) = \int\frac{d\mathbf{p}}{(2\pi)^2}\bar\chi_N(\mathbf{p}+\mathbf{k})\hat A_-(\mathbf{p}+\mathbf{k})\hat v(\mathbf{k}+\mathbf{p})u_N(\mathbf{p}) \times[1+\hat g^{[k+1,N]}_\omega(\mathbf{p})\hat W^{(0;2)(k)}(\mathbf{p})][1+\hat g^{[k+1,N]}_\omega(\mathbf{k})\hat W^{(0;2)(k)}(\mathbf{k})]. \quad (157)$$

For $0\le k\le N$, we define $\mathcal{L}H^{a(1;1)(k)}_{T,\varepsilon;\omega,\sigma}(\mathbf{k}) = H^{a(1;1)(k)}_{T,\varepsilon;\omega,\sigma}(\mathbf{k})$ for such terms; since $|\mathbf{k}|$ is fixed by hypothesis, $|\mathbf{p}+\mathbf{k}|\le C\gamma^N$ a bound for (157) is

$$|v|_\infty\gamma^{-k}\gamma^{-(N-k)}\left[1+C\sum_{j=k}^N\gamma^{-(j-k)}\right][1+C\gamma^{-(N-k)}] \le C\gamma^{-k}\gamma^{-(N-k)}. \quad (158)$$

On the other hand, for $k<0$, $\mathcal{L}\hat H^{a(1;1)(k)}_{T,\varepsilon;\omega,\sigma}(\mathbf{k}) = \hat H^{a(1;1)(k)}_{T,\varepsilon;\omega,\sigma}(0)$ and

$$\hat H^{a(1;1)(k)}_{T,\varepsilon;\omega,\sigma}(0) = \sum_{\omega'}D_{\omega'}(\mathbf{k})\int\frac{d\mathbf{p}}{(2\pi)^2}\bar\chi_N(\mathbf{p})\hat A_-(\mathbf{p})\hat v(\mathbf{p})(\partial_{\omega'}u_N)(\mathbf{p}) \times[1+\hat g^{[k+1,N]}_\omega(\mathbf{p})\hat W^{(0;2)(k)}(\mathbf{p})][1+\hat g^{[k+1,N]}_\omega(\mathbf{k})\hat W^{(0;2)(k)}(\mathbf{k})]. \quad (159)$$

Such an integral is zero. Indeed, we have

$$\hat A_-(\mathbf{p}) = \frac{\nu_N\hat v^2(\mathbf{p})D_{-\omega}(\mathbf{p})}{D^2_\omega(\mathbf{p})-\nu_N^2\hat v^2(\mathbf{p})D^2_{-\omega}(\mathbf{p})} = \frac{\nu_N\hat v^2(\mathbf{p})D_{-\omega}(\mathbf{p})}{D^2_\omega(\mathbf{p})}\sum_{p\ge0}\left(\frac{\nu_N\hat v(\mathbf{p})D_{-\omega}(\mathbf{p})}{D_\omega(\mathbf{p})}\right)^{2p} \stackrel{def}{=} \sum_{p\ge0}\hat A_{p,-}(\mathbf{p}). \quad (160)$$

Under a rotation of an angle ϑ, we have:

$$\hat A_{p,-}(\mathbf{p}^*) = e^{-i\omega\vartheta(4p+3)}\hat A_{p,-}(\mathbf{p}), \qquad \hat W^{(0;2)(k)}_{p',\omega,\sigma}(\mathbf{p}^*) = e^{-i\omega\vartheta(2p'-1)}\hat W^{(0;2)(k)}_{p',\omega,\sigma}(\mathbf{p}) \quad (161)$$

and therefore, since $(4p+4+2p')>0$, taking $\vartheta:\vartheta(4p+4+2p')<2\pi$, the integral (159), with $\hat A_-(\mathbf{p})$ and $\hat W^{(0;2)(k)}_{\omega,\sigma}(\mathbf{p})$ replaced by $\hat A_{p,-}(\mathbf{p})$ and $\hat W^{(0;2)(k)}_{p',\omega,\sigma}(\mathbf{p})$ respectively, is zero.

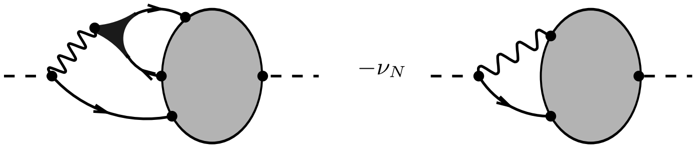


Fig. 11 Graphical representation of $H^{b(1;1)(k)}_{T,\varepsilon;\omega,\omega',\sigma}$

3. The contraction of T_1 with all and three fields contracted with the same kernel $\hat W^{(0;4)(k)}_{\omega,\underline{\sigma}}$; and the contraction of T_-. They are:

$$\begin{aligned}
&\hat H^{b(1;1)(k)}_{T,\varepsilon;\omega,\sigma}(\mathbf{k})\\
&\quad=\sum_{\bar\omega}\sum_{i,j=k}^{N}\int\frac{d\mathbf{p}\,d\mathbf{q}}{(2\pi)^4}\,\bar\chi_N(\mathbf{p})\hat A_\varepsilon(\mathbf{p})D_{\bar\omega}(\mathbf{p})\hat S^{(i,j)}_{\bar\omega,\omega}(\mathbf{q}+\mathbf{p},\mathbf{q})\\
&\qquad\times\hat g_\omega(\mathbf{p}+\mathbf{k})\hat W^{(0;4)(k)}_{\underline{\sigma}}(\mathbf{k}+\mathbf{p},\mathbf{q}+\mathbf{p},\mathbf{q})\\
&\qquad+\int\frac{d\mathbf{p}}{(2\pi)^2}\bar\chi_N(\mathbf{p})\hat A_\varepsilon(\mathbf{p})\hat v(\mathbf{p})D_{-\omega}(\mathbf{p})\\
&\qquad\times\hat g_\omega(\mathbf{p}+\mathbf{k})\hat W^{(1;2)(k)}_{\underline{\sigma}}(\mathbf{k}+\mathbf{p},\mathbf{p}).
\end{aligned}\tag{162}$$

It has a bound as (115) times a further factor

$$|v|_\infty\cdot\sum_{j=k}^{N}|g^j_\omega|_1\le C\gamma^{-k}.\tag{163}$$

For $k<0$, we have $H^{b(1;1)(k)}_{T,\varepsilon;\omega,\sigma}(0)=0$. This follows using (160), (161) and

$$\hat A_+(\mathbf{p})=\frac{\hat v(\mathbf{p})D_\omega(\mathbf{p})}{D^2_\omega(\mathbf{p})-\nu_N^2\hat v^2(\mathbf{p})D^2_{-\omega}(\mathbf{p})}\stackrel{def}{=}\sum_{p\ge0}\hat A_{p,+}(\mathbf{p}),\tag{164}$$

$$\begin{aligned}
&\hat A_{p,+}(\mathbf{p}^*)=e^{-i\omega\vartheta(4p+1)}\hat A_{p,+}(\mathbf{p}),\\
&\hat W^{(0;4)(k)}_{p',\underline{\sigma}}(\mathbf{p}^*)=e^{-i\omega\vartheta(2p'-2)}\hat W^{(0;4)(k)}_{p',\underline{\sigma}}(\mathbf{p}),\\
&D_{\bar\omega}(\mathbf{p}^*)\hat S^{(i,j)}_{\bar\omega,\omega}(\mathbf{q}^*+\mathbf{p}^*,\mathbf{q}^*)=e^{-i\omega\vartheta}D_{\bar\omega}(\mathbf{p})\hat S^{(i,j)}_{\bar\omega,\omega}(\mathbf{q}+\mathbf{p},\mathbf{q}).
\end{aligned}\tag{165}$$

This completes the proof. □

Appendix A: Bounds for the Δ Function

Because of the symmetry $\hat S^{(i,j)}_{\omega,\omega'}(\mathbf{p},\mathbf{q})=\hat S^{(j,i)}_{\omega,\omega'}(\mathbf{q},\mathbf{p})$, we will only concern the case $i\ge j$. A bound for $\hat S^{(i,j)}_{\omega,\omega'}$ can be obtained by explicit computation, using that

$$f_i(\mathbf{k})(1-\chi_N^{-1}(\mathbf{k}))=-\delta_{i,N}(1-f_N(\mathbf{k}))\stackrel{def}{=}-\delta_{i,N}u_N(\mathbf{k}).$$

1. For $i = j = N$,

$$\begin{aligned}
&\hat{U}^{(N,N)}_{\omega}(\mathbf{q}+\mathbf{p},\mathbf{q})\\
&\quad=\left[u_N(\mathbf{q})\frac{f_N(\mathbf{q}+\mathbf{p})}{D_\omega(\mathbf{q}+\mathbf{p})}-u_N(\mathbf{q}+\mathbf{p})\frac{f_N(\mathbf{q})}{D_\omega(\mathbf{q})}\right]\bar{\chi}_N(\mathbf{p})\\
&\quad=\frac{u_N(\mathbf{q})f_N(\mathbf{q}+\mathbf{p})}{D_\omega(\mathbf{q})D_\omega(\mathbf{q}+\mathbf{p})}\bar{\chi}_N(\mathbf{p})D_\omega(\mathbf{p})+\frac{f_N(\mathbf{q})}{D_\omega(\mathbf{q})}[u_N(\mathbf{q})-u_N(\mathbf{q}+\mathbf{p})]\bar{\chi}_N(\mathbf{p})\\
&\qquad+\frac{u_N(\mathbf{q})}{D_\omega(\mathbf{q})}[f_N(\mathbf{q}+\mathbf{p})-f_N(\mathbf{q})]\bar{\chi}_N(\mathbf{p}).
\end{aligned}\tag{166}$$

Therefore we obtain:

$$\begin{aligned}
\hat{S}^{(N,N)}_{\omega,\omega'}(\mathbf{q}+\mathbf{p},\mathbf{q}) \stackrel{def}{=} &-\delta_{\omega,\omega'}\bar{\chi}_N(\mathbf{p})\frac{u_N(\mathbf{q})f_N(\mathbf{q}+\mathbf{p})}{D_\omega(\mathbf{q})D_\omega(\mathbf{q}+\mathbf{p})}\\
&+\bar{\chi}_N(\mathbf{p})\frac{f_N(\mathbf{q})}{D_\omega(\mathbf{q})}\int_0^1 d\tau\,(\partial_{\omega'}u_N)(\mathbf{q}+\tau\mathbf{p})\\
&+\bar{\chi}_N(\mathbf{p})\frac{u_N(\mathbf{q})}{D_\omega(\mathbf{q})}\int_0^1 d\tau\,(\partial_{\omega'}f_N)(\mathbf{q}+\tau\mathbf{p}).
\end{aligned}\tag{167}$$

2. For $j < N$, using also that $u_N(\mathbf{q})f_j(\mathbf{q}) \equiv 0$ (the support of the two function is disjoint) we have

$$\begin{aligned}
\hat{U}^{(N,j)}_{\omega}(\mathbf{q}+\mathbf{p},\mathbf{q}) &= -\bar{\chi}_N(\mathbf{p})u_N(\mathbf{q}+\mathbf{p})\frac{f_j(\mathbf{q})}{D_\omega(\mathbf{q})}\\
&= -\bar{\chi}_N(\mathbf{p})[u_N(\mathbf{q}+\mathbf{p})-u_N(\mathbf{q})]\frac{f_j(\mathbf{q})}{D_\omega(\mathbf{q})}.
\end{aligned}\tag{168}$$

Hence:

$$\hat{S}^{(N,j)}_{\omega,\omega'}(\mathbf{q}+\mathbf{p},\mathbf{q}) = \bar{\chi}_N(\mathbf{p})\frac{f_j(\mathbf{q})}{D_\omega(\mathbf{q})}\int_0^1 d\tau\ (\partial_{\omega'}u_N)(\mathbf{q}+\tau\mathbf{p}).\tag{169}$$

3. For $i, j < N$, we have $\hat{U}^{(i,j)}_{\omega} \equiv 0$.

By inspection, since $|\partial_\omega f_j| \le C\gamma^{-j}$ as well as $|\partial_\omega u_N| \le C\gamma^{-N}$, we obtain that $\partial^m_{\mathbf{p}}\partial^n_{\mathbf{q}}\hat{S}^{(N,j)}_{\omega,\omega'}(\mathbf{p},\mathbf{q})$ is not identically zero if one between $i = N$; $\mathbf{q} : f_j(\mathbf{q}) \neq 0$ and $\mathbf{p} : |\mathbf{p}| \le 2\gamma^N$; in this case we obtain

$$|\partial^m_{\mathbf{p}}\partial^n_{\mathbf{q}}\hat{S}^{(N,j)}_{\omega,\omega'}(\mathbf{p},\mathbf{q})| \le C_{m,n}\gamma^{-N(1+m)-j(1+n)}.\tag{170}$$

The above bounds allow to obtain

$$\int \frac{d\mathbf{p}d\mathbf{q}}{(2\pi)^4}|\partial^m_{\mathbf{p}}\partial^n_{\mathbf{q}}\hat{S}^{(N,j)}_{\omega,\omega'}(\mathbf{p},\mathbf{q})| \le C'_{m,n}\gamma^{N(1-m)+j(1-n)}\tag{171}$$

from which the former of (104) follows. The analysis for $P^{(N,j)}_{\bar{\omega},\omega}$ is similar:

$$\begin{aligned}
\hat{Q}^{(N,j)}_{\omega}(\mathbf{q}+\mathbf{p},\mathbf{q}) &= \bar{\chi}_N(\mathbf{p})u_N(\mathbf{p}+\mathbf{q})\hat{\chi}_j(\mathbf{q})\\
&= \bar{\chi}_N(\mathbf{p})[u_N(\mathbf{p}+\mathbf{q})-u_N(\mathbf{q})]\hat{\chi}_j(\mathbf{q})
\end{aligned}\tag{172}$$

from which the latter of (104) follows.

Appendix B: Solution of the Closed Equation

By inserting (135) into (132), in the limit $N \to \infty$ we obtain

$$\partial_\omega S_{\omega,\sigma}(\mathbf{x}) = \delta(\mathbf{x}) + \lambda A_-(\mathbf{x}) S_{\omega,\sigma}(\mathbf{x}) \tag{173}$$

whose solution is

$$S_{\omega,\sigma}(\mathbf{x}) = \exp\left\{\lambda \int d\mathbf{z}\, [g_\omega(\mathbf{x}-\mathbf{z}) - g_\omega(\mathbf{z})] A_-(\mathbf{z})\right\} g_\omega(\mathbf{x}). \tag{174}$$

By (136), we first consider

$$\int \frac{d\mathbf{p}}{(2\pi)^2} e^{-i\mathbf{x}\cdot\mathbf{p}} \hat{g}_\omega(\mathbf{p}) \hat{v}(\mathbf{p}) a(\mathbf{p}) = \int \frac{d\mathbf{p}}{(2\pi)^2} F(\mathbf{p}) \frac{e^{-i\mathbf{x}\cdot\mathbf{p}}}{(p_0 + i\omega s(\mathbf{p}) p_1)(p_0 + i\omega p_1)} \tag{175}$$

where

$$s(\mathbf{p}) = \frac{1+\nu\hat{v}(\mathbf{p})}{1-\nu\hat{v}(\mathbf{p})}, \qquad F(\mathbf{p}) = \frac{\hat{v}(\mathbf{p})}{\nu\hat{v}(\mathbf{p}) - 1}.$$

Indeed (175) is well defined for $\mathbf{x} = 0$: we can rewrite it separating the two domains $|\mathbf{p}| \le 1$ and $|\mathbf{p}| > 1$. The integral on the latter is absolutely convergent, since the decay of $F(\mathbf{p})$ is faster than any power. The integral of the former can be written as

$$F(0) \int_{|\mathbf{p}|\le 1} \frac{d\mathbf{p}}{(2\pi)^2} \frac{1}{(p_0 + i\omega s p_1)(p_0 + i\omega p_1)} + R \tag{176}$$

where R is again an absolutely convergent integral; the first integral can be written as

$$\begin{aligned} &\int_{|\mathbf{p}|\le 1} \frac{d\mathbf{p}}{(2\pi)^2} \frac{1}{(p_0 + i\omega s p_1)(p_0 + i\omega p_1)} \\ &\quad = -\int_{|\mathbf{p}|\le 1} \frac{d\mathbf{p}}{(2\pi)^2} \frac{1}{(i\omega p_1 + s p_0)(i\omega p_1 + p_0)} \\ &\quad = -\int_{p_0^2 + s^2 p_1^2 \le 1} \frac{d\mathbf{p}}{(2\pi)^2} \frac{1}{(p_0 + i\omega p_1)(p_0 + i\omega s p_1)} \end{aligned} \tag{177}$$

hence, the above integral also equals

$$\int \frac{d\mathbf{p}}{(2\pi)^2} \frac{\chi(p_0^2 + p_1^2 \le 1) - \chi(p_0^2 + s^2 p_1^2 \le 1)}{(p_0 + i\omega s p_1)(p_0 + i\omega p_1)} \tag{178}$$

which is absolutely convergent since the support of $\chi(p_0^2 + p_1^2 \le 1) - \chi(p_0^2 + s^2 p_1^2 \le s^2)$ does not contain a neighborhood of the origin.

Now we discuss (175) for $\mathbf{x} \neq 0$. It can be written as $H_0 + H_1 + H_2$, for

$$H_0 = F(0) \int \frac{d\mathbf{p}}{(2\pi)^2} \frac{e^{-i\mathbf{x}\cdot\mathbf{p}}}{(p_0 + i\omega s p_1)(p_0 + i\omega p_1)},$$
$$H_1 = \int \frac{d\mathbf{p}}{(2\pi)^2} \frac{[F(\mathbf{p}) - F(0)]e^{-i\mathbf{x}\cdot\mathbf{p}}}{(p_0 + i\omega s p_1)(p_0 + i\omega p_1)}, \tag{179}$$
$$H_2 = \int \frac{d\mathbf{p}}{(2\pi)^2} F(\mathbf{p}) \frac{e^{-i\mathbf{x}\cdot\mathbf{p}}}{(p_0 + i\omega p_1)} \left[\frac{1}{(p_0 + i\omega p_1)} - \frac{1}{(p_0 + i\omega s(\mathbf{p}) p_1)} \right].$$

By straightforward computation, H_0 is given by

$$\frac{1}{2\pi(1-\nu)(s-1)} \int_0^{+\infty} \frac{dq_1}{q_1} [e^{-[|x_0|c + i x_1 \omega \mathrm{sgn}(x_0)]q_1} - e^{-[|x_0| + i x_1 \omega \mathrm{sgn}(x_0)]q_1}]$$
$$= \frac{1}{4\pi\nu} \ln \frac{x_0 + i\omega x_1}{x_0 s + i\omega x_1}, \tag{180}$$

while both H_1 and H_2 are vanishing as $\mathbf{x} \to \infty$. Indeed H_1 can be written as

$$\int_{|\mathbf{p}| \le N} \frac{d\mathbf{p}}{(2\pi)^2} [F(\mathbf{p}) - F(0)] \frac{e^{-i\mathbf{x}\cdot\mathbf{p}}}{(p_0 + i\omega s p_1)(p_0 + i\omega p_1)}$$
$$+ \int_{|\mathbf{p}| \ge N} \frac{d\mathbf{p}}{(2\pi)^2} F(\mathbf{p}) \frac{e^{-i\mathbf{x}\cdot\mathbf{p}}}{(p_0 + i\omega s p_1)(p_0 + i\omega p_1)}$$
$$+ F(0) \int_{|\mathbf{p}| \ge N} \frac{d\mathbf{p}}{(2\pi)^2} \frac{e^{-i\mathbf{x}\cdot\mathbf{p}}}{(p_0 + i\omega s p_1)(p_0 + i\omega p_1)}. \tag{181}$$

The second and third term are convergent integral, and each of them can be chosen small than $\frac{\varepsilon}{3}$ for N large enough; the first integral is vanishing as $\mathbf{x} \to \infty$. A similar argument holds for H_3.

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